L. T. McQuiston '38

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PLANE AND SPHERICAL TRIGONOMETRY

WITH STEREOGRAPHIC PROJECTIONS

BY

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PREFACE

It is the aim of the authors in this text to teach logarithmic computation along with the principles of trigonometry. Many students gain some knowledge of the theory of logarithms, but few appreciate their value as a labor-saver in computing. In engineering schools, especially, it is complained that students after their study of logarithms are unable to compute with them in courses in professional subjects. The authors feel that the course in trigonometry, with its excellent opportunities for logarithmic computation, is the place to remedy this weakness. With this idea in mind, logarithmic computation has been developed gradually and extended over the entire book. Plenty of problems of a varied character have been supplied for practice and illustration. The importance, in acquiring speed and accuracy, of systematic arrangement of work and of making out a form for the computation in advance before opening tables, is persistently emphasized.

Chapter I is devoted to explaining the theory of logarithms and to numerical computations — including the evaluation of exponential expressions — involving only the use of the table of *Logarithms of Numbers*. In succeeding chapters, along with the theoretical trigonometry, the other logarithmic tables are introduced and discussed as opportunity arises for their use.

The functions are defined for the general angle. The variations of the functions and their graphs are carefully discussed. In connection with the derivation of the formulae a large number of identities and transformations are given for drill purposes. The haversine is used in many places throughout the text — but alternate solutions are given for those who do not use the

Haversine Table. Chapters on trigonometric equations, inverse functions, and complex numbers are put at the end of the plane trigonometry.

Spherical Trigonometry is treated more fully than in most texts. Several methods are given for solving the different cases of the spherical triangle, and the particular usefulness of each method is pointed out. Many solved problems and forms for computation are shown. A complete chapter is devoted to the applications of trigonometry to navigation. The subject is concluded with two chapters on Stereographic Projection as a means of solving spherical triangles graphically.

The authors are greatly indebted to Dr. Louis Serle Dederick, Assistant Professor, U.S. Naval Academy, for his careful and detailed criticism of their manuscript and for many valuable suggestions.

The authors are also indebted to Mrs. W. W. Hendrickson for permission to make use of Professor Hendrickson's "Notes on Stereographic Projection."

PREFACE TO REVISED EDITION

A REVISION has been made to embody certain changes which class room experience at the U. S. Naval Academy indicated advisable. Considerable rearrangement of the original material has been made, some explanations condensed and others expanded. A new chapter on the functions of acute angles has been inserted after Chapter II, new diagrams and problems have been added, particularly in the chapters treating spherical trigonometry, and notation used in the study of alternating currents has been introduced in the chapter on complex numbers.

The authors are indebted to the Department of Mathematics, U. S. Naval Academy, for many suggestions and to the publishers for their coöperation.

J. A. B.

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TRIGONOMETRY

PART I—PLANE TRIGONOMETRY

CHAPTER I

LOGARITHMS

1. The work required in making many numerical computations, particularly those involving multiplication, division, involution (raising to a power) and evolution (extraction of a root), is considerably lessened by the use of Tables of Logarithms. Trigonometry gives rise to many problems of this sort. In the present chapter the theory of logarithms and their use in numerical computation is introduced, and throughout this text Trigonometry and Logarithmic Computation will be developed together.

2. Definition of a Logarithm.—The **Exponent** by which a number B must be affected to produce a number N is called the logarithm of N to the base B.

Thus, if $B^L = N$, we write $\log_B N = L$, which is read "The logarithm of N to the base B is L."

These two equations express the same relation between N and L and must at all times be associated in the definition of a logarithm. It is most important to understand that a logarithm is an exponent and hence obeys the laws of exponents. The student should be able to pass readily from the logarithmic notation to the exponential and from the exponential to the logarithmic. As illustrations,

$3^2 = 9$,	whence	$\log_3 9 = 2$;
$5^3 = 125$,	whence	$\log_5 125 = 3$;
$10^{1.5} = \sqrt{1000}$,	whence	$\log_{10}\sqrt{1000} = 1.5$.

N is sometimes called the "anti-logarithm" of L, or the "number corresponding" to L. Thus, we write, anti-$\log_3 2 = 9$, which is read "the anti-logarithm of 2 to the base 3 is 9."

EXERCISE I

1. Find the value of $\log_2 8$, $\log_3 81$, $\log_5 3125$, $\log_a(\sqrt{a^5})$.

2. Find the value of $\log_4 2$, $\log_{27} 3$, $\log_2(1/8)$, $\log_{10}(0.01)$.

3. Write anti-$\log_{10} 3$, anti-$\log_3 2$, anti-$\log_{10}(-4)$.

4. Find the base for each of the following: $\log_x 9 = 2$, $\log_y 9 = 2/3$, $\log_z 144 = 4$.

5. Write $\log_7 7$, $\log_{10} 10$, $\log_b b$, $\log_{10} 1$, $\log_b 1$.

6. What is the value of $\log_1 1$? $\log_1 2$? $\log_1 9$? Why cannot 1 be used as the base for a system of logarithms?

7. What is the value of $\log_{-3} 9$? $\log_{-3} 27$? Why cannot a negative number be used as a base for a system of logarithms?

8. What is the value of $\log_3(-9)$? $\log_3(-27)$? $\log_B(-a)$? Show that negative numbers have no real logarithms for positive bases.

9. Show that $\log_b a \cdot \log_c b \cdot \log_a c = 1$.

10. Show that $b^{\log_b x} = x$.

11. Solve for x

(a) $2 \log_2 x + \log_2 4 = 1$.

(b) $\log_3 x + 3 \log_3 9 = 4$.

3. There are two systems of logarithms in general use, although the possible number is unlimited. One system is called the Natural or Napierian, in which the base is $e = 2.71828\ldots$; the other is the Common or Briggs' system, in which the base is 10.

Generally, computations are performed with the Common System and it is with this that we shall be concerned. Throughout this text wherever the base is omitted it is to be understood that it is 10. Thus, log 0.1572 means $\log_{10} 0.1572$.

By employing only the definition of a logarithm and the elementary laws of exponents it is possible to find the logarithm of any number to the base 10, to any desired degree of accuracy.

To find the logarithm of any number N to the base 10 let $\log N = \frac{P}{Q}$.

Then, $10^{\frac{P}{Q}} = N$; or, $10^P = N^Q$; or, $N^Q 10^{-P} = 1$.

If $N^{Q_1} \cdot 10^{-P_1} = a_1$ where $a_1 < 1$

and $N^{Q_2} \cdot 10^{-P_2} = a_2$ where $a_2 > 1$

then $N^{(Q_1+Q_2)}\ 10^{-(P_1+P_2)} = a_1\, a_2$.

The product $a_1 \cdot a_2$ lies between a_1 and a_2; that is, it is nearer to 1 than one of these quantities and perhaps nearer than either. This process may be repeated indefinitely. Thus to find log 3 to three decimals:

$3^2 \cdot 10^{-1} = 0.9$ ..... (1)

Squaring (1), $3^4 \cdot 10^{-2} = 0.81$ ..... (2)

" (2), $3^8 \cdot 10^{-4} = 0.656$ ..... (3)

" (3) $3^{16} \cdot 10^{-8} = 0.430$ ..... (4)

Multiplying (2) and (4), $3^{20} \cdot 10^{-10} = 0.349$ ..... (5)

" (5) by 3, $3^{21} \cdot 10^{-10} = 1.046$ ..... (6)

" (1) and (6), $3^{23} \cdot 10^{-11} = 0.941$ ..... (7)

" (6) and (7), $3^{44} \cdot 10^{-21} = 0.985$ ..... (8)

" (6) and (8), $3^{65} \cdot 10^{-31} = 1.030$ ..... (9)

" (8) and (9), $3^{109} \cdot 10^{-52} = 1.014$ ..... (10)

" (8) and (10), $3^{153} \cdot 10^{-73} = 0.999$ ..... (11)

But (11) may be written (approximately $3^{153} \cdot 10^{-73} = 1$),

$3^{153} = 10^{73}$ or $3 = 10^{\frac{73}{153}}$.

Therefore, $\log 3 = \frac{73}{153} = 0.477$.

4. Laws of Operation with Logarithms.—Since the logarithm is an exponent, its laws of operation must conform to those of exponents.

Thus, we know from the theory of exponents

$$a^x \cdot a^y = a^{x+y}.$$
$$a^x \div a^y = a^{x-y}.$$
$$(a^x)^y = a^{xy}.$$
$$\sqrt[y]{a^x} = a^{\frac{x}{y}}.$$

In terms of logarithms:

(1) *The logarithm of the product of two positive numbers is equal to the sum of the logarithms of the two numbers.*

Let us take two numbers and express each as a power of the base B.

$$N = B^{L_1},\ M = B^{L_2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

or
$$\log_B N = L_1,\ \log_B M = L_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

then
$$N \cdot M = B^{L_1} \cdot B^{L_2} = B^{L_1+L_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

But (3) is in the form of a number $(N \cdot M)$ equal to a base (B) affected by an exponent (L_1+L_2) and can, therefore, be written

$$\log_B N \cdot M = L_1 + L_2.$$

Substituting from (2) we have

$$\log_B N \cdot M = \log_B N + \log_B M.$$

This proof can readily be generalized to cover the product of any number of positive numbers, merely by taking them two at a time. Thus, *the logarithm of a product is the sum of the logarithms of the several factors.*

(2) *The logarithm of the quotient of two numbers is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Let us use the same numbers as in the previous demonstration.

$$\text{Then } \frac{N}{M} = \frac{B^{L_1}}{B^{L_2}} = B^{L_1-L_2};$$

or,
$$\log_B \frac{N}{M} = L_1 - L_2 = \log_B N - \log_B M.$$

(3) *The logarithm of a number affected by an exponent is equal to the product of the exponent and the logarithm of the number.*

Thus, if $N = B^L$, $\log_B N = L$,

$N^C = (B^L)^C = B^{LC}$;

or, $\log_B N^C = LC = C \cdot \log_B N$.

This demonstration applies to any real value of C whether it is positive or negative, integral or fractional.

EXERCISE II

Given $\log 2 = 0.30103$, $\log 3 = 0.47712$, $\log 5 = 0.69897$, $\log 7 = 0.84510$.

Illustrations.—**1.** Find $\log 4$.

$\log 4 = \log 2^2 = 2 \log 2 = 2 \times 0.30103 = 0.60206$.

2. Find $\log 35$.

$\log 35 = \log 5 \times 7 = \log 5 + \log 7 = 0.69897 + 0.84510 = 1.54407$.

3. Find $\log 75$.

$\log 75 = \log 5^2 \times 3 = 2 \log 5 + \log 3 = 1.87506$.

4. Find $\log 400$.

$\log 400 = \log 10^2 \times 2^2 = 2 \log 10 + 2 \log 2 = 2 + 0.60206 = 2.60206$.

Exercises involving only the given logarithms

Find:

1. $\log 30$.

2. $\log 98$.

3. $\log \frac{10}{7}$.

4. $\log \frac{162}{49}$.

5. $\log 525$.

6. $\log \sqrt[5]{3}$.

7. $\log \sqrt[4]{28}$.

8. $\log (2^{\frac{2}{3}} \times 21^{\frac{1}{2}})$.

9. $\log \sqrt[10]{324}$.

10. $\log \frac{3^{\frac{6}{7}}}{5^{\frac{1}{7}}}$.

11. $\log 36^{\frac{1}{3}}$.

12. $\log 42^5$.

13. $\log 2^8$.

14. $\log 48^{\frac{1}{4}}$.

15. $\log (\frac{14}{5})^{\frac{1}{4}}$.

16. $\log \frac{3^{\frac{5}{3}}}{\sqrt{7}}$.

5. Characteristic and Mantissa.—From algebra we know that

$10^0 = 1.$ $\qquad 10^{-1} = \frac{1}{10} = 0.1.$

$10^1 = 10.$ $\qquad 10^{-2} = \frac{1}{100} = 0.01.$

$10^2 = 100.$

$10^3 = 1000.$ $\qquad 10^{-3} = \frac{1}{1000} = 0.001.$

$10^4 = 10000.$ $\qquad 10^{-4} = \frac{1}{10000} = 0.0001.$

These equations are in the form $B^L = N$; we shall now put them in the form $\log_B N = L$.

$\log 1 = 0.$ $\qquad \log 0.1 = -1.$

$\log 10 = 1.$ $\qquad \log 0.01 = -2.$

$\log 100 = 2.$ $\qquad \log 0.001 = -3.$

$\log 1000 = 3.$ $\qquad \log 0.0001 = -4.$

A glance at the tables above will show that the logarithm of a number greater than 1 is positive, while that of a number between 0 and 1 is negative.

The tables used in computation are logarithms of positive numbers. However, we shall show later that this is no barrier to performing computation involving negative numbers with these tables.

From the table above it is evident that the logarithm of any number between

1 and 10 is 0+ a decimal.
10 " 100 " 1+ a decimal.
100 " 1000 " 2+ a decimal, etc.

1 and 0.1 (lies between 0 and −1, or) is −1+a decimal.
0.1 and 0.01 (lies between −1 and −2, or) is −2+a decimal.
0.01 and 0.001 (lies between −2 and −3, or) is −3+ a decimal, etc.

The integral part of the logarithm is called the **characteristic** while the fractional part, which is always expressed decimally, is called the **mantissa.**

6. Rules for Characteristic and Mantissa.—By the method indicated in Art. 3 and by other more exact methods it is possible to compute as close approximations as may be desired of the logarithms of numbers.

These values have been arranged in the form of tables. Tables have been constructed giving the logarithms to various numbers of decimals. A five-place table is one in which the logarithms are given to five decimals.

The tables of logarithms give only the mantissa or the fractional part of the logarithm, consequently the characteristic must always be supplied. The mantissa does not depend upon the magnitude of the number, but merely upon the sequence of the figures.

To illustrate,

$$\log 3.1416 = 0.49715$$
$$\log 31.416 = \log 10 \times 3.1416 = 1 + 0.49715$$
$$\log 314.16 = \log 10^2 \times 3.1416 = 2 + 0.49715$$
$$\log 3141.6 = \log 10^3 \times 3.1416 = 3 + 0.49715$$

$$\log 0.31416 = \log 10^{-1} \times 3.1416 = -1 + 0.49715$$
$$\log 0.031416 = \log 10^{-2} \times 3.1416 = -2 + 0.49715$$
$$\log 0.0031416 = \log 10^{-3} \times 3.1416 = -3 + 0.49715$$

Thus, if

$$\log_{10} N = L, \text{ then } \log_{10} 10^n \cdot N$$
$$= n \log_{10} 10 + \log_{10} N = n + \log_{10} N,$$

where n is any number whatever.

From the above we notice that the log 0.31416 is $-1+0.49715$ which is -0.50285; this is the actual logarithm of 0.31416, and -1.50285 is the actual logarithm of 0.031416, etc. This form, however, is inconvenient, as it would involve subtracting each time we took the logarithm of a number which is less than 1.

Hence it is customary to replace the -1 by $9-10$, the -2 by $8-10$, etc., and write such logarithms in the form $9.49715-10$, $8.49715-10$, etc. It must be clearly understood, however, just what is meant: i.e., $-10+9.49715$, $-10+8.49715$, etc.

In laying out the work for computation the -10 is sometimes not written, but in that case the student must understand that the -10 is intended.

For speed in computation it is customary to determine the characteristic from rules, which are committed to memory.

Referring again to the last table given in Art. 5 we find that the characteristic of a number equal to or greater than 1 and less than 10 is 0; equal to or greater than 10 and less than 100 is 1; equal to or greater than 100 and less than 1000 is 2; etc.

RULE.—*The characteristic of the logarithm of a number greater than* 1 *is* positive *and is one less than the number of digits to the left of the decimal point.*

Considering the numbers less than 1, we observe that a number equal to or greater than 0.1 and less than 1, has the characteristic -1 or $9.-10$; equal to or greater than 0.01 and less than 0.1 has -2 or $8.-10$; equal to or greater than 0.001 and less than 0.01 has -3 or $7.-10$; etc.

RULE.—*The characteristic of the logarithm of a decimal fraction is always* negative *and is equal to the number of the place occupied by the first significant figure,* (i.e., first figure different from 0).

In practice, as mentioned above, the negative is always replaced by $9.-10$, $8.-10$, etc. In this case the usual rule is to *subtract the number of zeros preceding the first significant figure from* 9 *and use this with the* -10 *as the characteristic.*

EXERCISE III

Determine the characteristic in each of the following:

1.	27.341.	**6.**	0.896.	**11.**	567.81.
2.	15700.	**7.**	321472.	**12.**	0.0003894.
3.	1.275.	**8.**	0.00000000001.	**13.**	358946.
4.	0.00349.	**9.**	1.0000237.	**14.**	0.0037581.
5.	1.0004.	**10.**	0.03754928.		

USE OF TABLES OF LOGARITHMS

7. To Find the Logarithm of a Number.—First determine and write down the characteristic.

If the number whose logarithm is desired is made up of four figures, find the first three in the left hand column of the table and on the line with this in the column headed by the fourth figure will be found the mantissa. If the given number is of less than four figures, make it one of four by adding zeros; that is, if the number is 17 look up 17.00, if the number is 0.0019 look up 0.001900, etc.

In case the number consists of five figures, it is necessary to perform what is called interpolation.

Let us illustrate this: Suppose we desire the logarithm of 37.983. The characteristic is 1; the table will give us,

$$\text{mantissa log } 3799 = 0.57967$$
$$\text{mantissa log } 3798 = \underline{0.57955}$$

Tabular difference between mantissas 0.00012.

The mantissa of the log of 37.983 is between the two given mantissas. In practice we assume that the change in the logarithm is proportional to the change in the number. Since an increase of 1 in the number increased the mantissa 0.00012, an increase of 0.3 in the number will increase the mantissa 0.3 as much or $0.00012 \times 0.3 = 0.000036$ or 0.00004. As a matter of practice we say 0.3 of $12 = 3.6$ or 4. Hence,

$$\log 37.983 = 1.57959.$$

The last figure in a logarithm is always an approximation, and as a result it is useless to carry logarithms beyond the number of places given in the table. It is for this reason that we used 4 instead of 3.6; had this been 3.4 or less we should have called it 3; when it is 3.6 or greater we assign the value 4; had it been 3.5, we should still have assigned the value 4, as most computers employ the rule of taking the even number when the interval splits evenly. Thus, 3.5 we call 4; 4.5 we call 4; 5.5, 6; 6.5, 6; etc.

In case the number consists of more than five figures, two courses are open; the mantissa of the first four figures may be found as explained above, and to this may be added the product of the tabular difference by the remaining figures considered as a decimal number; or the number may be made into one of five digits by approximating on the last figure. Sufficient accuracy will be obtained by the second method for ordinary work.

In the table are to be found "difference tables," which facilitate the work. They are sufficiently accurate for most practical purposes and should be used. Thus in the illustration just given one would find a small table headed "12" on the page with the given mantissa. This table consists simply of 0.1 of 12, 0.2 of 12, 0.3 of 12, etc., and on the line with 1 gives 1, with 2 gives 2, with 3 gives 4, etc. Hence, in finding the mantissa of 37983 we would note the tabular difference as 12, seek the small table, and opposite 3 secure 4, the required difference. We shall illustrate this principle in the following exercises; in addition various minor suggestions will be brought out by way of illustration.

8. **Illustrative Examples.—1.** Find log 38.756.

Characteristic 1; mantissa of 3876 is 58838; mantissa 3875 is 58827; diff. is 11. In "diff" table 11, on the line with 6 is the number 7; adding 7 to 58827 gives 58834. *This should all be done mentally.* Hence, log 38.756 is 1.58834.

2. Find log 0.0000000000038756.

The mantissa here is just the same as in **1** i.e., 58834. The rule given for finding the characteristic of the logarithm of a number less than 1 was to subtract the number of zeros from 9, and put -10 after the mantissa. In this unusual case however we have more than nine zeros. Here we would use **19.**-20 and write

$$\log 0.0000000000038756 = 8.58834 - 20.$$

3. Find log $\sqrt[3]{0.087458}$.

From Article 4, we have,

$$\log_B N^C = C \log_B N.$$

Applied here $\log \sqrt[3]{0.087458} = \log (0.087458)^{\frac{1}{3}} = \frac{1}{3} \log (0.087458)$.

$$\log 0.087458 = 8.94180 - 10.$$

We cannot conveniently divide this logarithm by 3 because of the -10; hence we write it $28.94180 - 30$, which can be so divided. As a result we have,

$$\tfrac{1}{3} \log 0.087458 = 9.64727 - 10.$$

Note that the actual division would have given $9.64726\frac{2}{3} - 10$. As previously explained we retain only 5 places in the mantissa.

4. Find log of $\sqrt[5]{0.0853273^2}$.

$$\log \sqrt[5]{0.0853273^2} = \tfrac{2}{5} \log 0.0853273.$$

Characteristic $8.-10$; in place of 853273 use 85327, mantissa 0.93109;

$$\log 0.0853272 = 8.93109 - 10$$

$$\underline{\qquad\qquad 2}$$

$$17.86218 - 20$$

$$5 \,|\, \underline{47.86218 - 50}$$

$$9.57244 - 10$$

Therefore, $\log \sqrt[5]{0.0853273^2} = 9.57244 - 10$.

It will be noted that the multiplication by 2 was performed before the division by 5. This is a general practice to avoid multiplying the error in case the division is not exact.

EXERCISE IV

Verify the following:

1. log 3798.5 = 3.57961.
2. log 3798.2 = 3.57957.
3. log 3798.3 = 3.57959.
4. log 3.7987 = 0.57963.
5. log 0.0053829 = 7.73101 − 10.
6. log 17.986 = 1.25493.
7. log 239.75 = 2.37976.
8. log 0.75328 = 9.87696 − 10.
9. log 1255.2 = 3.09873.
10. log 0.0042767 = 7.63111 − 10.

Find logs of the following:

11.	3426.	**17.**	0.0075329.	**23.**	7506.7.
12.	71.	**18.**	10.817.	**24.**	900080.
13.	8.	**19.**	271.83.	**25.**	12.023.
14.	7.854.	**20.**	9.9999.	**26.**	0.12203.
15.	3.1416.	**21.**	102.01.	**27.**	0.002174.
16.	30.473.	**22.**	0.36413.		

9. To Find the Anti-logarithm.—We shall next take up the converse of finding the logarithm of a given number; i.e., finding the anti-log or number corresponding to a given logarithm. The operation throughout is just the reverse of what we have already discussed and we shall illustrate it with several examples, in each case explaining the various steps.

A logarithm, the mantissa of which is given exactly in the table.

1. Find the number having 2.39217 for its logarithm.

In the table we find the exact mantissa 39217. On the line with this and in the number column at the left we find 246; above 39217 and in the same column we find 7. The given characteristic is 2 and as a result we have *three* significant figures to the left of the decimal point.

Therefore, the required number is 246.70.

This is often called the anti-log of 2.39217.

2. Find anti-log of $7.86747 - 10$.

In the table we find the exact mantissa 86747. On the line with it and at the left we find 737; at the top is 0. Since the characteristic is $7. - 10$ there are two zeros to the right of the decimal point preceding the first significant figure of the anti-log.

Therefore, the required anti-log is 0.0073700.

Where the mantissa as given does not occur in the table.

In this case it will occur between two successive mantissas in the table. Hence, our problem is to find these and interpolate in a manner which is the reverse of the one we used in seeking the logarithm of a given number.

3. Find anti-log of 2.92238.

The mantissa 92238 is found between the two mantissas 92236 and 92241 of the table to which correspond respectively the numbers 8363 and 8364 as explained in **1.** The number we seek is between these; i.e., it is greater than 8363 and less than 8364. The tabular difference (that is, the difference between the two successive mantissas) is 5; this difference corresponds to a difference of 1 in the numbers. The given mantissa 92238 is 2 greater than the lesser 92236. Hence, the number required (disregarding for the moment the location of the decimal point which depends wholly upon the characteristic) is $8363\frac{2}{5}$ or 83634. Locating the decimal point from the characteristic 2, which by the rule is one less than the number of digits to the left of the decimal point, we have anti-log of 2.92238 is 836.34.

As explained before, the use of the table of proportionate parts is recommended. In the present case we seek the table headed 5. In the right hand column is found the number 2 and corresponding to it in the left hand column there is a choice of 3 or 4.

This is an illustration also of the slight inaccuracy involved in the use of the table; in this case our result could be written either as 836.33 or 836.34. In general, for all parts of the table, five-place accuracy cannot be obtained with a five-place table. When greater accuracy is desired a table must be used with a sufficient number of places to insure it.

In general, never carry the result beyond the number of places given in the table, i.e., never have more than five significant figures in an anti-log obtained from a five-place table.

4. Find the anti-log of $8.55099-10$.

In the table we find the mantissa 55096 as the next smaller than 55099. The tabular difference at this point is 12; our difference is 3. In the table of proportionate parts headed 12, and in the right hand column we seek a 3; there is none and we use the 2; in which case our result is 0.035562 (See Art. 7).

EXERCISE V

Find the anti-logs of the following:

1.	2.49751.	**5.**	3.97833.	**9.**	5.30103.
2.	1.82537.	**6.**	$7.09263-10$.	**10.**	$9.47713-10$.
3.	6.15932.	**7.**	3.20079.	**11.**	2.49715.
4.	8.32796.	**8.**	0.32002.	**12.**	$6.00703-10$.

10. Computation.—A careful arrangement of the work is necessary for speed and accuracy in computation. The directions given in this article will help the student to arrange his work in a systematic manner.

First write down the formulae by means of which the unknown quantities are to be determined, and then make out a blank form for the problem. *This form should be complete.* Once the form is finished it should no longer be necessary to consider the problem or the formulae by which it is to be worked; it should merely be necessary to fill in the spaces left blank. A few general remarks as to the nature of this form follow.

(*a*) Instead of writing log 932.57 = 2.96969 write, 932.57 log 2.96969.

(*b*) All given data and blank spaces for unknown data, which are to be computed, should be in the first column at the left.

(*c*) On the horizontal line with each quantity are written all logarithms, and only those logarithms, which pertain to that quantity.

(*d*) A column of work is usually devoted to the computation of each unknown.

(*e*) Every step necessary in the work should be clear from the form.

It is practically impossible to state general laws for making the outline and exception may be found even for those given. However, these act as a guide. A study of the forms used in the text, both blank and completed, will aid the student in making satisfactory ones for himself.

Example 1.—Find $y = 2727.5 \times 0.37375 \times 51.101$.

By (1) Art. 4

$$\log y = \log 2727.5 + \log 0.37375 + \log 51.101.$$

The form for this computation follows:

2727.5	log
0.37375	log
51.101	log
$y =$	log

As an exercise fill in this form and compute y. *Ans.* 52093.

Example 2.—Find the value of

$$x = [\sqrt{732.95}\ (8.90172)\ (0.0954)^{\frac{3}{5}}]^2.$$

$$\log x = 2\ [\tfrac{1}{2} \log 732.95 + \log 8.90172 + \tfrac{3}{5} \log 0.0954].$$

732.95	log	$\frac{1}{2}$ log
8.9017		log
0.0954	log	$\frac{3}{5}$ log
$x = 3463.4$	log 3.53950	$\frac{1}{2}$ log

There are a few points in this illustration which we shall emphasize.

The second column of logarithms is made up of the logarithms after they have been affected by the exponent. The first column is made up of the original logarithms in the cases where they are to be affected. The logarithm which is obtained by the addition is not the logarithm of the answer but is one-half of it because the quantity in the bracket is to be squared. All of this is taken care of in making out the form. The form should be completed in this manner before opening the tables.

11. Co-logarithms. We have shown the logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers; i.e.,

$$\log \frac{N}{M} = \log N - \log M. \qquad (1)$$

In order to obviate the necessity of subtraction it is customary to use the co-logarithm (written co-log). The co-logarithm, often called the arithmetical complement, is the logarithm of the reciprocal of the number. Thus, since the reciprocal of N is $\frac{1}{N}$,

$$\text{co-log } N = \log \frac{1}{N} = \log 1 - \log N = 0 - \log N = -\log N,$$

$$\text{or co-log } N = -\log N.$$

Hence (1) can be written

$$\log \frac{N}{M} = \log N - \log M = \log N + \text{co-log } M.$$

In the case of more than one divisor it is usually much simpler to perform the computation by means of the co-logarithms.

To find the co-logarithm of a number we subtract the logarithm from 0 as indicated above. This 0 can be written as $10.-10$ or $20.-20$, etc., as is most convenient. Thus, co-log 84.321 is

		10.00000	−10
84.321	log	1.92594	
84.321	co-log	8.07406	−10.

Usually this is done mentally by beginning at the left and subtracting each figure of the logarithm from nine excepting the last significant figure at the right which is taken from ten.

Thus log of 842.27 is 2.92546 while its co-log is 7.07454; 0.073264, log 8.86489−10, co-log 1.13511; 0.00053926, log 6.73180−10, co-log 3.26820.

12. Illustrative Examples Involving Co-logarithms.

1. Evaluate,

$$X=\left[\frac{3769875}{973.21\times 0.0047}\right]^{\frac{1}{5}}$$

$\log X=\frac{1}{5}$ [log 3769875+co-log 973.21+co-log 0.0047].

The form for this is given below.

		3769875.	log
		973.21	co-log
		0.0047	co-log
Ans.	$X=$	log	5 log

As a second illustration we shall perform a computation involving negative numbers. The computation is made as if the numbers were positive but the negative sign is carried through the work in the manner shown.

2. Find the value of $\sqrt[3]{\frac{ab^2}{c}}$, given $a=23.975$; $b=-5.793$; $c=-179.82$.

a,	23.975		log	1.37976
b,	−5.793	log (−) 0.76290	2 log	1.52580
c,	−179.82		co-log (−)	7.74516−10
Ans.	−1.6478	log (−) 0.21691	3 log (−)	0.65072

In this case b is negative and therefore log b is marked (−), 2 log b is not because b has been raised to an even power and as

a result is positive. Adding the column of logs corresponds to multiplying the quantities; hence an odd number of negative signs will cause the final result to be $(-)$; an even number of $(-)$ signs would mean that the result was a positive number and hence the $(-)$ would not occur before the sum logarithm. In this example there is but one $(-)$ in the last column; hence the sum of the logarithms also carries this $(-)$. We then take $\frac{1}{3}$ of this logarithm which means that we are securing the cube root of the number represented by this log. The cube root of a negative number is itself negative; hence the $(-)$ is brought down and the corresponding number or anti-log is negative. Had we attempted to take an even root of the number represented by our sum logarithm it would have been an imaginary number.

Care must be taken not to confuse this $(-)$ which is merely a convention for handling negative numbers, with the negative characteristic. This $(-)$ indicates that the number corresponding to the logarithm is negative and indicates nothing else.

EXERCISE VI

Find by logs the value of:

1. $\dfrac{2.9275\times 34.278}{505.92}$.
2. $\dfrac{0.0031593\times 684.82}{0.00096548}$.
3. 5th root of 29736800.
4. 4th root of (293.76×3271.9).
5. $\dfrac{(-0.00019158)^{\frac{2}{5}}\times\sqrt{68.002}}{0.27556}$.
6. $\dfrac{\sqrt[5]{-0.0030121}\times 1954.7}{(-0.84132)^{8}\times\sqrt[4]{17959}\times(-560.63)^{\frac{2}{7}}}$.
7. $\sqrt[5]{\dfrac{a^2b}{c^3d^4}}$ when $a=-4.9679$. $b=-500.02$. $c=-0.70675$. $d=9.7086$.

8. Find the value of $P=\sqrt{\frac{(s-a)\ (s-b)\ (s-c)}{s}}$

where $2s=a+b+c$. $a=6.8235$; $b=5.2063$; $c=3.1628$.

9. Same as **8**, when $a=65.43$; $b=58.26$; $c=49.23$.

10. Find the area of a triangle of which the sides are **10.385 ft.**, 11.353 ft., and 17.486 ft.

$K=\sqrt{s(s-a)\ (s-b)\ (s-c)}$, s as in example **8**.

13. Computations Involving Sums and Differences.

1. Find value of $\frac{a^2c^{\frac{1}{2}}}{b+D}$ when $D=\frac{a^{\frac{1}{2}}}{b}$, for

$a=23.722$; $b=571.17$; $c=0.03218$.

$a=23.722$	log	$\frac{1}{2}$ log	2 log
$b=571.17$		co-log	
$D=$		log	
$b+D=$			co-log
$c=0.03218$	log		$\frac{1}{2}$ log
$\frac{a^2c^{\frac{1}{2}}}{b+D}=$			log

This is a complete form for the example; a slight study of it will suggest how to handle a problem involving addition or subtraction and will further emphasize the value of making out a form previous to the use of the table.

2. A common example is to find one leg of a triangle when the hypotenuse and the other leg are given.

Given a right triangle, hypotenuse c and leg a; to find the other leg b.

$$b=\sqrt{c^2-a^2}=\sqrt{(c+a)\ (c-a)}.$$

Form:

	$c=$		
	$a=$		
	$c+a=$		log
	$c-a=$		log
Ans.	$b=$	log	2 log

3. Find value of $\sqrt[3]{\frac{a^{\frac{1}{3}}\,b}{a^2-b}}$, where $a=7.5328$; $b=6384$.

a =	7.5328	$\frac{1}{3}$ log	0.29232
a^2 =	56.744	2 log	1.75392
b =	6384	log	3.80509
a^2-b =	−6327.256	co-log (−)	6.19878−10
		3 log (−)	0.29619
	−1.2552	log (−)	0.09873

4. Find value of $\sqrt{\frac{b}{a^3}+\sqrt[3]{a^2c}}$ when $a=735.91$; $b=0.198$; $c=27$. Let $X=\frac{b}{a^3}$ and $Y=\sqrt[3]{a^2c}$.

Form.

a = 735.91	log	co-log	3 co-log	2 log
b = 0.198			log	
c = 27.				log
				3 log
Y =				log
X =			log	
$X+Y$ =				
Ans. =	log			
	$\frac{1}{2}$ log			

EXERCISE VII

Find values of following:

1. $\dfrac{9999.5\times(-0.0077764)}{(-28421)}$.

2. $\dfrac{(-29.285)\times(-0.07562)}{411.44\times(-38.456)}$.

3. $\sqrt[3]{-90.}$

4. $\sqrt[3]{-10.974}$.

5. $(-19683)^{-\frac{5}{9}}$.

6. $\left(\dfrac{\sqrt{a\cdot b\cdot c}+1}{d\cdot e\cdot f}\right)^3$ where $a=-339.48$, $b=-0.019168$. $c=0.0029167$, $d=0.67233$. $e=0.0067229$, $f=29165$.

7. $\sqrt[3]{\dfrac{(-0.95048)^5\,(8473)^{\frac{4}{3}}}{(-2080.9)\,\sqrt[6]{0.0572}}}$.

8. $\dfrac{\sqrt[5]{-0.003012}\times 1.955}{(-0.843)^8\times\sqrt[4]{17959}\times(-560.6)^{\frac{2}{7}}}$.

9. $\dfrac{(a^3+b^{\frac{1}{3}})\,(b-c^{-2})}{a^{-1}\,b^2\,c}$, where $a=3.3542.$
$b=7.1954.$
$c=0.00982.$

10. $\sqrt[3]{\dfrac{a^4}{b-c^{\frac{1}{3}}}}$ when $a=-0.079438.$
$b=0.00038795.$
$c=0.0000062193.$

11. $\sqrt[5]{\dfrac{a^{-\frac{2}{3}}}{b^2-c^2}}$ when $a=-0.005678.$
$b=52.132.$
$c=3.5961.$

12. $\sqrt[5]{\dfrac{a^2\cdot b\cdot c^3}{d\cdot e}}+f$ when $a=-0.032713.$
$b=643.68.$
$c=-0.99985.$
$d=68432.$
$e=-0.000835.$
$f=0.0005.$

13. The hypotenuse of a right triangle is equal to 957.38 and one leg is 125.49. Find the other leg.

14. One leg of a right triangle is 873.9 and the other is 537.24. Find the hypotenuse.

14. Whenever it is necessary to multiply or divide a logarithm by a number of several digits it is best to do this by means of logarithms.

For example, to find the value of $(29.37)^{2.9183}$,

Let $X=(29.37)^{2.9183}$,

then $\log X=2.9183 \log 29.37$.

Remembering that logarithms are themselves numbers, we can proceed and write

$$\log \text{ of } (\log X) = \log 2.9183 + \log \text{ of } (\log 29.37).$$

The logarithm of a logarithm is called the "log log" and is written llog. Hence, our form for this example would be

2.9183		log
29.37	log	llog
Ans.	log	llog

If the number whose llog is desired is less than 1 its log will be negative and in this case the negative log should be used.

For example, find the llog of 0.0483.

$\log 0.0483 = 8.68395 - 10 = -1.31605,$

$\text{llog } 0.0483 = \log -1.31605 = (-)0.11926.$

Care should be taken to distinguish between these minus signs.

Illustrations:

1. $x^{2.983} = 0.07936$; find x.

$$2.983 \log x = \log 0.07936,$$

$$\log x = \frac{\log 0.07936}{2.983},$$

and $\text{llog } x = \text{llog } 0.07936 + \text{co-log } 2.983.$

0.07936	$\log 8.89960 - 10 = -1.10040$	$\text{llog}(-)$	0.04155
2.983		co-log	$9.52535 - 10$
$x = 0.42767$	$\log 9.63111 - 10 = -0.36889$	$\text{llog}(-)$	$9.56690 - 10$

The student must not jump to the conclusion that because a llog occurs in a problem that the sum of the logarithms is necessarily the llog of the required unknown. The algebraic steps first given above must be gone through with care, and from the final equation the information necessary for the form will be obtained.

2. Find x when $(3.7)^x = 0.8173$.

$x \log 3.7 = \log 0.8173$,

$$x = \frac{\log 0.8173}{\log 3.7}.$$

$\log x = \operatorname{llog} 0.8173 + \operatorname{collog} 3.7$.

0.8173	log 9.91238 − 10 = −0.08762	llog(−)8.94260 − 10	
3.7	log 0.56820	co-llog	0.24550
$x = -0.15421$		log(−)9.18810 − 10	

3. Find x from the equation $x = \log_{0.0954} 87.543$.

$0.0954^x = 87.543$; $x \log 0.0954 = \log 87.543$.

$$x = \log_{0.0954} 87.543 = \frac{\log 87.543}{\log 0.0954},$$

$\log x = \operatorname{llog} 87.543 - \operatorname{llog} 0.0954$.

87.543	log 1.94223	llog	0.28830
0.0954	log 8.97955 − 10 = −1.02045	llog(−)	0.00879
$x = -1.9033$		log(−)	0.27951

EXERCISE VIII

Solve the following equations:

1. $20^x = 100$.

2. $5^{2x} = 317.46$.

3. $7^x = 100$.

4. $\log_x 2 = 0.69315$.

5. $5^{\frac{1}{x}} = 1.307$.

6. $1.05^x = (8.25)^{\frac{1}{x}}$.

7. $(0.9)^{\frac{1}{x^2}} = (4.7)^{-\frac{1}{3}}$.

8. $x^{0.35678} = 8$.

9. $\log_x 5.9723 = -0.47216$.

10. $\log_{7.829} x = 0.054391$.

11. $\log_7 0.057624 = x$.

12. $2x^{2.56} = 1.5724$.

13. $3^{x+y} = 2(5^x)$, $x - y = 1$.

14. $a = e^{cx}$, when $a = 375850$, $c = 1.5708$, $e = 2.7183$.

15. $(ax)^n = bx^m$, when $a = 0.74219$, $n = 0.92972$, $b = 7.9426$, $m = 1.89625$.

16. $(0.9x)^{89.32} = 75438.5$.

17. $(8732)^{-1.5327x} = 57.2987$.

18. $\log_x 87.543 = -17.932$.

19. $2^x 3^{x-2} = 0.5$.

20. $x = (0.96745)^{0.96745}$.

21. How long will it take a sum of money to double itself if put at 4% compound interest? This is represented by $(1.04)^x = 2$ where x is the number of years. Solve for x.

22. Given $a = 4.7549$, $b = 0.96745$, find $a \times b$, $\frac{a}{b}$, $\frac{\log a}{\log b}$, $\log a \cdot \log b$, and a^b.

15. Change of Base.—Logarithms to a base B may be converted into logarithms to any other base A as follows:

$$\log_A N = X, \qquad \text{or } A^X = N.$$

Taking the logarithm of both sides of this latter equality, to the base B, we have

$$\log_B A^X = X \log_B A = \log_B N,$$

which can be written

$$X = \frac{\log_B N}{\log_B A}, \qquad \text{or } \log_A N = \frac{\log_B N}{\log_B A}.$$

In other words we have the $\log_A N$ expressed as the quotient of two logarithms, both to the base B. Applying this formula in the case of e and 10 we have,

$$\log_e N = \frac{\log_{10} N}{\log_{10} e},$$

and

$$\log_{10} N = \frac{\log_e N}{\log_e 10}.$$

Let $\frac{1}{\log_e 10} = \log_{10} e = M$, then

$$M = 0.43429 \qquad \text{and } \frac{1}{M} = 2.3026.$$

Therefore, $\log_e N = 2.3026 \log_{10} N$.

$\log_{10} N = 0.43429 \log_e N$.

EXERCISE IX

1. Find $\log_e 7$.
2. Find $\log_e 99$.
3. Find $\log_e 37$.
4. Make a table of the natural logarithms of the numbers from 1 to 10.
5. Find $\log_3 0.7$.
6. Find $\log_{27} 0.00328$.
7. Find $\log_{0.88} 99324$.
8. Find $\log_{21} 0.09827$.
9. Find $\log_{37} 5.9$.
10. Find $\log_8 27569$.
11. Find $\log_{0.09} 389.47$.
12. Find x if $\log_e x = 4.83721$.
13. Find x if $\log_e x = 3.94216$.
14. Find x if $\log_e x = 0.31743$.

CHAPTER II

ANGLES

16. Trigonometry.—If any three parts of a plane triangle—one at least being a side—are known, the triangle is completely determined. Trigonometry enables us to solve the triangle, that is, to compute the unknown parts, by means of certain ratios called *trigonometric functions.* In its original sense trigonometry is *that branch of mathematics which treats of the solution of triangles,* but in a broader sense it includes *mathematical investigations by means of trigonometric functions.*

Plane Trigonometry treats of plane triangles, Spherical Trigonometry of the triangles formed by the arcs of great circles on a spherical surface.

17. Angles.—In geometry an angle has been defined as the opening between two intersecting lines. In trigonometry the word "angle" receives an extension of meaning.

Let any line, coinciding, in its original position, with OA, Fig. 1, revolve about the point O until it takes the position OB.

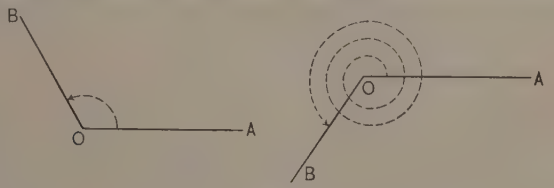


FIG. 1

We may say that this revolution of the line has generated the angle AOB. Thus in any position of its revolution it has generated an angle, and as there is no limit to the number of times it may revolve about O, there is no limit to the size of the angle. We shall call the point O, the *origin;* the revolving line, *the radius vector*; the position OA, the *initial position;* the position OB, the *terminal position.*

As an illustration of the meaning of the word angle consider the minute hand of a clock. At three o'clock consider it as being in its initial position; at 3:15 it will have generated a right angle; at 3:45, three right angles; at 4, four right angles; at 4:30, six right angles; at 6 o'clock, 12 right angles, etc

Thus an angle is measured by the amount of revolution a radius vector makes about an origin, in going from its initial to its terminal position.

18. Positive and Negative Angles.—If the radius vector moves in the opposite direction to that of the hands of a watch or clock when the observer looks at its face, the angle generated is called positive; if the radius vector rotates with the hands of a clock the angle is said to be negative. Thus we say the angle is positive if the rotation is counter-clockwise and negative if the rotation is clockwise.

19. Systems of Angular Measurement.—There are two systems of angular measurement in common use, namely, sexagesimal measure and radian measure.

In sexagesimal measure the units are,

the degree $= 1/90$ of a right angle;
the minute $= 1/60$ of a degree;
the second $= 1/60$ of a minute.

They are respectively denoted $1°$, $1'$, $1''$.

The unit of radian measure is the *radian*, which is defined as a *central angle subtended by an arc equal in length to the radius of the circle.*

Another system called the centesimal system was proposed in connection with the metric system and it is used to some extent in France. In it the units are,

the grade $= 1/100$ part of a right angle;
the minute $= 1/100$ part of a grade;
the second $= 1/100$ part of a minute.

They are respectively denoted 1^g, $1^`$, $1^{``}$.

20. Conversion of Units.

1. Theorem.—*The radian is a constant angle equal to* $\frac{180^\circ}{\pi}$.

From Geometry

$$\frac{\text{Radian}}{\text{2 rt. angles}} = \frac{R}{\pi R} = \frac{1}{\pi}.$$

$$\text{Radian} = \frac{\text{2 rt. angles}}{\pi} = \frac{180^\circ}{\pi}.$$

This relation

$$\pi \text{ radians} = 180^\circ$$

FIG. 2

enables us to pass from one system of angular measurement to the other.

To change degrees to radians we multiply the number of degrees by $\frac{\pi}{180}$; to change radians to degrees we multiply the number of radians by $\frac{180}{\pi}$. For example: $x = 43°27'45''$ $= 43.4625° = 43.4625 \times \frac{\pi}{180}$ radians.

43.4625	log 1.63812
π	log 0.49715
180	co-log 7.74473
$x = 0.75858$ radians	log 9.88000

Again, $\frac{5\pi}{6}$ radians $= \frac{5\pi}{6} \times \frac{180^\circ}{\pi} = 150°$.

EXERCISE X

1. Give the sexagesimal measure of the following

$$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{11\pi}{6}.$$

2. Change to circular measure 12°, 15°, 75°, 45°, 181.2°, 315°, 24.7°.

3. Find the circular measure of $36°27'43''$, $118°03'51''$, $219°37'12''$, $335°20'18''$, $40°34'27''$.

4. Change from circular to sexagesimal measure the following: 2.816, 3, 1.782, 0.941, 0.06781.

5. Show that 1 radian $= 57.2958° = 3437.75' = 206265''$.

6. Show that $1° = 0.0174533$ radians, $1' = 0.00029089$ radians, $1'' = 0.000004848$ radians.

7. The difference between two angles of a triangle is $\pi/11$, their sum is $90°$; find the sexagesimal measure of each angle of the triangle.

8. The angles of a triangle form an arithmetical progression, and the number of degrees in the least is to the number of radians in the greatest as $60 : \pi$; find each angle in circular measure.

9. At what time between two and three o'clock are the hour and minute hands of a clock $90°$ apart? At what time are they $150°$ apart?

10. The unit of angular measurement used in the U. S. Field Artillery is the "mil," defined as the angle subtended by 1/6400 of a circumference. Show that a mil is approximately 1/1000 of a radian.

11. Find the value of a mil in sexagesimal measure.

21. Theorem.—*The radian measure of an angle at the center of a circle is equal to the intercepted arc divided by the radius.*

In Fig. 3 let AB be an arc equal in length to the radius OA, then $\angle AOB$ is a radian. Let $\angle AOC$ be the central angle intercepting the arc AC.

Then, $$\frac{\angle AOC}{AOB} = \frac{AC}{AB},$$

Therefore,

$$\angle AOC = \frac{AC}{R} \cdot \angle AOB = \frac{AC}{R} \text{ radians.}$$

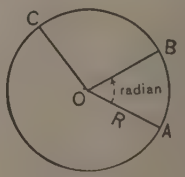


FIG. 3

Thus the number of radians in the angle can be found by dividing the intercepted arc by the radius of the circle. Sometimes this theorem is expressed by the formula

$$\text{arc} = \text{radius} \times \text{angle in radians.}$$

Example.—A car which is travelling with uniform speed in a circular track of radius 100 yds. is observed to traverse an arc subtending an angle of 60° in one minute. Find its speed in miles per hour.

Solution: The arc traversed in one minute $100 \times 60° \times \frac{\pi}{180°}$ yds.

Therefore, speed $= 100 \times 60° \times \frac{\pi}{180°} \times 60 \times \frac{1}{1760}$

$= 3.57$ miles per hour (approximately).

EXERCISE XI

1. The bob of a 33-inch pendulum describes an arc of 4 inches. Find, to the nearest minute, the angle through which it swings.

2. A geographic mile is equal to an arc of 1′ on the earth's equator. Find its value in feet if the earth's diameter is 7924 miles.

3. Two places on the same meridian are 168 miles apart, find their difference in latitude. Use diameter of earth as 7924 miles.

4. Find the length of a belt stretched around two wheels of radius 6 ft. and 1 ft., respectively, the distance between their centers being 10 ft.

5. If the angle subtended by the sun's diameter at the eye of an observer is 32′, find approximately the diameter of the sun if its distance from the observer is 92,900,000 miles.

In this example the diameter of the sun is really the chord of the arc of which the observer's eye is the center, but differs

relatively little from the arc in length. Hence, we may assume that the arc gives the approximate length of the chord.

6. If the diameter of the moon is 2163 miles and it subtends an angle of 31′7″ at the eye of an observer on the earth, find the approximate distance from the earth to the moon.

7. At what distance will a church spire, 156 feet high, subtend an angle of 40′ at the eye of an observer, both standing on the same plane?

NOTE.—Assume the spire to form the arc of a circle which has its center at the eye of the observer.

8. The semi-circumference of a circle is equal to its diameter plus a certain arc. Find the central angle subtended by that arc.

9. A horse is tethered to a stake at the corner of a field where the boundaries intersect at an angle of 75°. How long must the rope be so that the horse can graze over half an acre?

10. Find the length in feet of an arc of 3″ on the earth's equator.

11. An enemy battery, range 6000 yds., subtends an angle of 12 mils. How many yards of front does it occupy? (See Prob. 10. Ex. X.)

CHAPTER III

FUNCTIONS OF AN ACUTE ANGLE

22. Trigonometric Functions of an Acute Angle.—If A and B are the acute angles and a, b, and c the sides of a right triangle as shown in Fig. 4, then the trigonometric functions of the angle A are defined by the following equations:

$$\text{sine of } A = \frac{\text{side opposite}}{\text{hypotenuse}} = a/c,$$

$$\text{cosine of } A = \frac{\text{side adjacent}}{\text{hypotenuse}} = b/c,$$

$$\text{tangent of } A = \frac{\text{side opposite}}{\text{side adjacent}} = a/b,$$

$$\text{cotangent of } A = \frac{\text{side adjacent}}{\text{side opposite}} = b/a,$$

$$\text{secant of } A = \frac{\text{hypotenuse}}{\text{side adjacent}} = c/b,$$

$$\text{cosecant of } A = \frac{\text{hypotenuse}}{\text{side opposite}} = c/a.$$

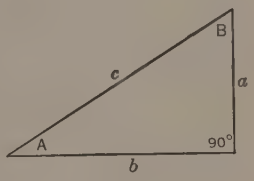


FIG. 4

These equations are usually written as follows:
$\sin A = a/c$, $\cos A = b/c$, $\tan A = a/b$, $\cot A = b/a$, $\sec A = c/b$, $\csc A = c/a$, and express general definitions, namely that: *In a right triangle,*

The sine of an acute angle is the side opposite divided by the hypotenuse.

The cosine of an acute angle is the side adjacent divided by the hypotenuse.

The tangent of an acute angle is the side opposite divided by the side adjacent.

The cotangent of an acute angle is the side adjacent divided by the side opposite.

The secant of an acute angle is the hypotenuse divided by the side adjacent.

The cosecant of an acute angle is the hypotenuse divided by the side opposite.

Example.—Construct an angle whose tangent is 0.75 and find the other functions of this angle.

If $\tan X = 0.75 = \frac{3}{4}$, a right triangle can be constructed with sides 3, 4, and 5 as in Fig. 5, with the angle X opposite the side 3. Applying the definitions of the functions to this angle X we have

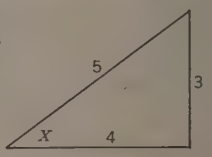


FIG. 5

$\sin X = \frac{3}{5} = 0.60$, $\quad \cot X = \frac{4}{3} = 1.33$,
$\cos X = \frac{4}{5} = 0.80$, $\quad \sec X = \frac{5}{4} = 1.25$,
$\tan X = \frac{3}{4} = 0.75$, $\quad \csc X = \frac{5}{3} = 1.67$.

23. Complementary Angles.—Since, in Fig. 4, $B = 90° - A$, let the student show:

$$\sin A = \cos (90° - A),$$
$$\tan A = \cot (90° - A),$$
$$\sec A = \csc (90° - A).$$

The sine and cosine are called cofunctions, similarly the tangent and cotangent, also the secant and cosecant.

Thus: *Any function of an acute angle is the cofunction of the complement of that angle.*

EXERCISE XII

1. The legs of a right triangle are 5 and 12, what are the functions of the smaller angle of the triangle?

2. The hypotenuse of a right triangle is 20 and a leg is 12, what are the functions of the included angle?

3. If $\sin Y = \frac{7}{25}$, construct the angle Y and find the other functions of Y.

4. If $\sec A = \frac{25}{20}$, construct the angle A and find the other functions of A.

5. If $\cos B = \frac{15}{17}$, construct the angle B and find the other functions of B.

6. If $\sin Y = \frac{7}{25}$, find the functions of $90° - Y$.

Find X given:

7. $\sin 4X = \cos 5X$.

8. $\tan (270° - X) = \cot 2X$.

9. $\sec (29° + X) = \csc (19° - 3X)$.

10. $\cos (3X - 12°) = \sin (4X + 33°)$.

24. Functions of 30°, 60°, 45°.—A 30°-angle can be constructed by drawing an altitude of an equilateral triangle as shown in Fig. 6. Let the side of the triangle be 2, then the length of the altitude is $\sqrt{3}$, and from the figure

$$\sin 30° = \frac{1}{2}, \qquad \cos 30° = \frac{\sqrt{3}}{2},$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}, \qquad \cot 30° = \sqrt{3},$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \qquad \csc 30° = \frac{2}{1} = 2.$$

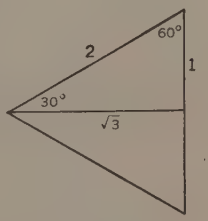


FIG. 6

Let the student verify the following:

$$\sin 60° = \frac{\sqrt{3}}{2}, \qquad \cos 60° = \frac{1}{2},$$

$$\tan 60° = \sqrt{3}, \qquad \cot 60° = \frac{\sqrt{3}}{3},$$

$$\sec 60° = 2, \qquad \csc 60° = \frac{2\sqrt{3}}{3}.$$

The functions of 45° can be obtained by applying the definitions to an acute angle of an isosceles right triangle and are:

$$\sin 45° = \cos 45° = \frac{\sqrt{2}}{2},$$

$$\tan 45° = \cot 45° = 1,$$

$$\sec 45° = \csc 45° = \sqrt{2}.$$

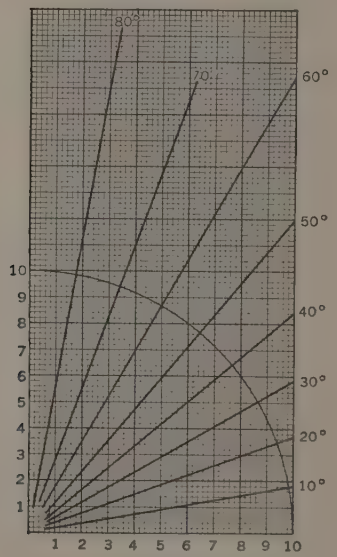


FIG. 7

EXERCISE

Figure 7 shows a quadrant of a circle of radius 10 units drawn on coördinate paper with angles marked off with a protractor. Let the student make a table of the sines and tangents of 0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80° by reading the values to the nearest tenth from this figure.

Degrees	Sine	Cosecant	Tangent	Cotangent	Cosine	Secant	
0	0.000	∞	0.000	∞	1.000	1.000	90
1	0.017	57.299	0.017	57.290	1.000	1.000	89
2	0.035	28.654	0.035	28.636	0.999	1.001	88
3	0.052	19.107	0.052	19.081	0.999	1.001	87
4	0.070	14.336	0.070	14.301	0.998	1.002	86
5	0.087	11.474	0.087	11.430	0.996	1.004	85
6	0.105	9.567	0.105	9.514	0.995	1.006	84
7	0.122	8.206	0.123	8.144	0.993	1.008	83
8	0.139	7.185	0.141	7.115	0.990	1.010	82
9	0.156	6.392	0.158	6.314	0.988	1.012	81
10	0.174	5.759	0.176	5.671	0.985	1.015	80
11	0.191	5.241	0.194	5.145	0.982	1.019	79
12	0.208	4.810	0.213	4.705	0.978	1.022	78
13	0.225	4.445	0.231	4.331	0.974	1.026	77
14	0.242	4.134	0.249	4.011	0.970	1.031	76
15	0.259	3.864	0.268	3.732	0.966	1.035	75
16	0.276	3.628	0.287	3.487	0.961	1.040	74
17	0.292	3.420	0.306	3.271	0.956	1.046	73
18	0.309	3.236	0.325	3.078	0.951	1.051	72
19	0.326	3.072	0.344	2.904	0.946	1.058	71
20	0.342	2.924	0.364	2.747	0.940	1.064	70
21	0.358	2.790	0.384	2.605	0.934	1.071	69
22	0.375	2.669	0.404	2.475	0.927	1.079	68
23	0.391	2.559	0.424	2.356	0.921	1.086	67
24	0.407	2.459	0.445	2.246	0.914	1.095	66
25	0.423	2.366	0.466	2.145	0.906	1.103	65
26	0.438	2.281	0.488	2.050	0.899	1.113	64
27	0.454	2.203	0.510	1.963	0.891	1.122	63
28	0.469	2.130	0.532	1.881	0.883	1.133	62
29	0.485	2.063	0.554	1.804	0.875	1.143	61
30	0.500	2.000	0.577	1.732	0.866	1.155	60
31	0.515	1.942	0.601	1.664	0.857	1.167	59
32	0.530	1.887	0.625	1.600	0.848	1.179	58
33	0.545	1.836	0.649	1.540	0.839	1.192	57
34	0.559	1.788	0.675	1.483	0.829	1.206	56
35	0.574	1.743	0.700	1.428	0.819	1.221	55
36	0.588	1.701	0.727	1.376	0.809	1.236	54
37	0.602	1.662	0.754	1.327	0.799	1.252	53
38	0.616	1.624	0.781	1.280	0.788	1.269	52
39	0.629	1.589	0.810	1.235	0.777	1.287	51
40	0.643	1.556	0.839	1.192	0.766	1.305	50
41	0.656	1.524	0.869	1.150	0.755	1.325	49
42	0.669	1.494	0.900	1.111	0.743	1.346	48
43	0.682	1.466	0.933	1.072	0.731	1.367	47
44	0.695	1.440	0.966	1.036	0.719	1.390	46
45	0.707	1.414	1.000	1.000	0.707	1.414	45
	Cosine	Secant	Cotangent	Tangent	Sine	Cosecant	Degrees

25. Tables of Natural Functions.—Tables, giving the approximate values of the trigonometric functions of various angles, have been constructed for use in computation by means of these functions. Such tables are called "Tables of Natural Functions." On the preceding page there is shown a table giving the values of the functions to three decimal places for every degree from 0° to 90°. The second column of this table contains the sines (reading down) of the angles at intervals of a degree from 0° to 45°, or the cosines (reading up) of the angles for every degree from 45° to 90°, since the cofunctions of complementary angles are equal. Similarly, the sixth column gives the cosines of angles from 0° to 45°, reading down, or the sines of angles from 45° to 90°, reading up. Thus to find the function of an angle between 0° and 45°, the number of degrees is found in the column at the left and the name of the function at the top of the table; to find the function of an angle between 45° and 90°, the number of degrees is read from the column at the right and the name of the function at the bottom of the table.

Example.—In a right triangle $a=27.3$ ft., $B=37°$; find the remaining parts of the triangle.

From Fig. 4: $\tan B=b/a$ and $b=a\times\tan B$;
$\sec B=c/a$ and $c=a\times\sec B$;
$A=90°-B=53°$.

From the table $\tan B=0.754$ and $\sec B=1.252$.

Thus, $b=27.3\times0.754=20.6$ ft.,
$c=27.3\times1.252=34.2$ ft.

26. Table 41, Bowditch.—The Bowditch Table 41 gives the values of the Natural Sines and Cosines to five decimals for angles at an interval of one minute from 0° to 90°. At the top of the page the degrees run from 0° to 44° and at the bottom from 45° to 89°. On each page there are two columns giving minutes from 0′ to 60′. The one on the left applies to the degrees listed at the top; the one on the right to those at the

bottom. Thus to find the Natural Sine (N sin) 44°25′ we find the 44° at the top and 25′ in the left column; on the line with the 25′ and under sine we obtain 0.69987. To find N cos 60°10′ we find the 60° at the bottom and in the column marked cos *at the bottom*, and on a line with 10′ on the right, we read 0.49748.

When the angle is not an exact number of minutes it becomes necessary to interpolate. In practice the same assumptions are used here as in the logarithm tables. To illustrate,

find N sin 14°40′11″.

N sin 14°41′ = 0.25348
N sin 14°40′ = 0.25320
Tab. Diff. 28

$$\frac{11}{60} \text{ of } 28 = \frac{308}{60} = 5.$$

Therefore, N sin 14°40′11″ = 0.25325.

On each page there are two columns of Proportional Parts; also the columns of functions are arranged in pairs. The Proportional Parts to the left are to be used with the left-hand columns of functions; the one to the right goes with the right-hand columns of functions. The number heading the column of Proportional Parts may not be exactly equal to the tabular difference; nevertheless, use it as it is an approximation for all the columns to which it pertains.

For interpolation the minute column of the table serves also as a seconds column. In the illustration find 11 seconds and on a line with it in the Proportional Parts column is 5.

When the value of a function is given and the angle desired, the process is the reverse of what has been given. As an illustration, if sin $\theta = 0.78932$, find θ.

N sin 52°8′ = 0.78944
N sin 52°7′ = 0.78926
Tab. Diff. 18

The sine of 52° is in the right-hand column; therefore, use the Proportional Parts column at the right. The given sine is 6 greater than 0.78926; hence seek a 6 in the Proportional Parts. There are several; in a case of this kind it is customary to take the one midway of the range. In this case we should obtain 20″.

Therefore, $\theta = 52°7'20''$.

Care must be used to ascertain from the tables whether the value of the function is increasing or decreasing, since in the former case we add the correction, while in the latter case we subtract it.

Find $N \cos 17°56'14''$.

$N \cos 17°57' = 0.95133$

$N \cos 17°56' = 0.95142$

Tab. Diff. 9

Our correction either from $\frac{14}{60}$ of 9 or Proportional Parts is 2. Here the 2 must be subtracted since the function decreases as the angle increases.

Hence $N \cos 17°56'14'' = 0.95140$.

EXERCISE XIII

Verify the following:

$N \sin 0°27'51'' = 0.00810$. $N \sin \theta = 0.89742$, $\theta = 63°49'12''$.

$N \cos 31°53'09'' = 0.84911$. $N \sin \theta = 0.13552$, $\theta = 7°47'18''$.

$N \cos 83°17'38'' = 0.11678$. $N \cos \theta = 0.43750$, $\theta = 64°3'20''$.

$N \sin 87°37'25'' = 0.99914$. $N \cos \theta = 0.01325$, $\theta = 89°14'26''$.

1. Find the value of the sum of

(a) the sines of the following angles:

(b) the cosines of the following angles:

$9°15'12''$; $60°40'55''$; $82°19'17''$; $43°19'10''$.

2. Given $\sin A = 0.18723$, $\sin B = 0.83627$, $\cos C = 0.37829$; $\cos D = 0.91356$, find the value of $A + B + C + D$.

3. Find $\sin A+\sin B$, $\sin (A+B)$, $\cos A+\cos B$, and $\cos (A+B)$ when $A=23°17'24''$, $B=58°14'10''$.

4. Find the remaining parts of the right triangle in which

(1) $a=17$, $A=69°$.
(2) $c=900$, $B=42°$.
(3) $b=8.5$, $A=17°$.
(4) $c=50$, $A=78°$.

5. Find the angles (to nearest degree) of the right triangle in which

(1) $a=5.3$, $c=12.7$.
(2) $c=68.7$, $b=29.4$.
(3) $a=3$, $b=4$.
(4) $a=55.6$, $b=21.9$.

6. Find the area of a regular dodecagon inscribed in a circle of radius 10 feet.

7. A ladder, 30 feet long, is placed against a building so as to make an angle of 67° with the ground. How far from the building is the lower end of the ladder?

8. Find to the nearest degree the angles of an isosceles triangle whose equal sides are 31 feet and whose base is 17 feet.

9. Find the length of the side of a regular polygon of 15 sides circumscribed about a circle of diameter 33 feet.

10. The height of a gable roof is 18 feet and its width is 38 feet, what is the pitch of the roof?

11. A tree 61 feet high casts a shadow 39 feet long on the level ground, how many degrees above the horizontal is the sun?

12. A body moves with a velocity of 30 feet per second along a line making 23° with the horizontal; find the horizontal and vertical components of the velocity. The component of the velocity is equal to the product of the velocity and the cosine

of the angle between the direction of the velocity and the direction of the component.

13. A balloon rises with a vertical speed of 10 feet per second while it encounters a horizontal wind of 12 miles per hour. In what direction does the balloon move and with what speed?

14. A force of 400 pounds acts at an angle of 36° with the horizontal. Find the horizontal and vertical components of the force. The definition of component is given in problem 12.

CHAPTER IV

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

27. Points and Real Numbers.—Let us consider a line such as XX' (Fig. 8) of indefinite length; and let us designate by O any fixed point on that line.

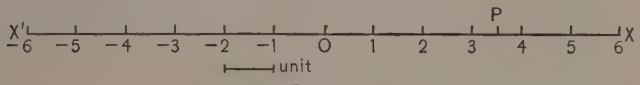


FIG. 8

We shall use O as a point from which to measure distances and shall call it the origin. Choose any length whatever as a unit and then lay it off on the line XX', beginning at O. Let us agree to lay off positive multiples of the unit to the right and negative multiples to the left of O. If we start now at O and apply our unit of length we shall locate 1, 2, 3, 4, 5, etc., on the right and -1, -2, -3, etc., on the left. Thus, the point at 3 units to the right represents the number 3; one at 3 units to the left represents the number -3. Conversely the number 3 represents a point on the line 3 units to the right of O while -3 represents a point on the line 3 units to the left of O.

If P is any point on the line the segment OP is also used to represent the number denoted by the point P; that is, if P is at 4, then OP is 4; while if P is half way between -2 and -3, OP is $-2\frac{1}{2}$.

28. Segments of the Line.—Wherever we wish to designate a segment of the line we shall employ two letters as AB; the segment is then considered to begin at A and end at B, i.e., to have the direction from A toward B. If this direction is toward the right the segment is positive; if toward the left it is negative. Hence

$$BA = -AB \quad \text{or} \quad BA + AB = 0$$

The *value* of a segment is determined simply by its length and direction; the value is defined as the number which would represent the end of the segment if the beginning of the segment were taken as the origin. Thus, in Fig. 9 if we represent the origin as is customary by O, then $OD=6$, $AB=1$, $AP_3=-1$, $BP_2=6$, $DP_3=-11$, etc.

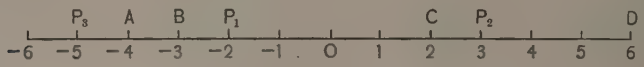


Fig. 9

If A, B, and C are any three points on a line then,

$$AC=AB+BC$$

This statement can easily be verified by trying the various arrangements of the three letters in a line.

29. Rectangular Coördinates.—A point in a plane may be located by giving the distances from two perpendicular lines in the plane as shown in Fig. 10. These lines are called the coördinate axes; XX' being the *X-axis* or *axis of abscissas*, YY' being the *Y-axis* or *axis of ordinates*.

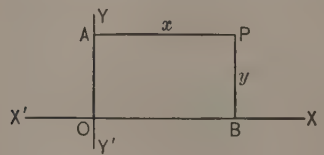


Fig. 10

Draw the line AP parallel to OX and intersecting OY at A; also draw BP parallel to OY and intersecting OX at B. Then AP (denoted by x) is called the *x-coördinate* or the *abscissa* of P and BP (denoted by y) is called the *y-coördinate* or the *ordinate* of P. The point P is designated by the symbol (x, y) and its position is determined by the directed segments x and y.

In discussing the location of a point on a line it was agreed to consider segments measured from left to right as positive and from right to left as negative. We shall adopt a similar convention for *vertical* segments; i.e., segments directed upward will be considered to be positive, those directed downward to be

negative. In general, segments not parallel to the coördinate axes will not have a definite sign attached to them. They will be treated as *positive* excepting where the two opposite directions along a straight line are considered, in which case either one can be called positive and the other negative.

The plane is divided by two axes into four parts, called *Quadrants*; they are numbered as shown in Fig. 11.

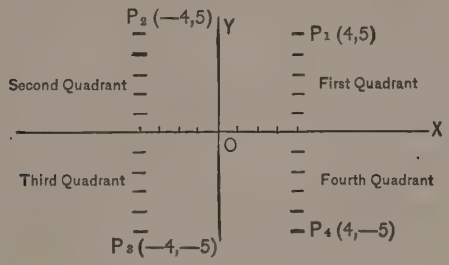


FIG. 11

In the first quadrant the abscissa and ordinate of any point are both positive; in the second, the abscissa is negative and the ordinate positive; in the third, both are negative; in the fourth, the abscissa is positive and the ordinate negative.

To locate a point as P_1 (4, 5) we first assume some length as a unit and then measure 4 units to the right and 5 units up. To locate P_2 $(-4, 5)$ measure 4 units to the left and 5 up; P_3 $(-4, -5)$, 4 units to the left and 5 down; P_4 $(4, -5)$, 4 units to the right and 5 down.

EXERCISE XIV

Locate and plot the following points:

1. (5, 0), (4, 3), (0, −5), (−3, 4), (−4, 3), $(\sqrt{10}, \sqrt{15})$, $(-\sqrt{21}, 2)$. These points have been so chosen that they lie on the circumference of a circle. Check results by drawing the circle. What is the radius?

2. (0, 8), (1, 15/2), (−2, 9), (−6, 11), (12, 2), (20, −2). These points should lie on a straight line.

30. Trigonometric Functions.—Associated with any angle there are six ratios which constitute the foundation of the study of Trigonometry. They are called either the trigonometric ratios, or the trigonometric functions, of the angle.

In order to make our discussion general, we shall illustrate with angles so generated that the terminal line is in each of the four quadrants (Fig. 12).

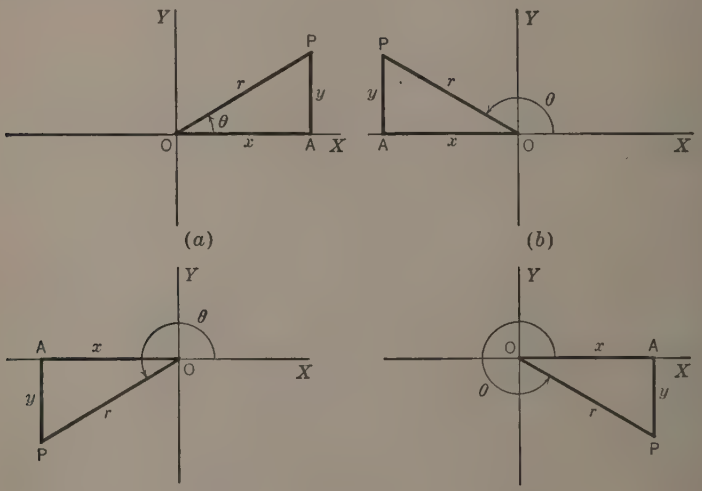


FIG. 12

Choose the point P on the terminal side of the angle and let r, called the *distance*, be the length OP. From P drop a perpendicular to the X-axis and designate by A its foot on this axis. Then, in each case P is a point whose abscissa is x and whose ordinate is y.

The six functions are defined as follows:

$$\textit{sine of angle } \theta \text{ (written } \textit{sin } \theta) = \frac{\text{ordinate}}{\text{distance}} = \frac{AP}{OP} = \frac{y}{r}.$$

$$\textit{cosine of angle } \theta \text{ (written } \textit{cos } \theta) = \frac{\text{abscissa}}{\text{distance}} = \frac{OA}{OP} = \frac{x}{r}.$$

$$\textit{tangent of angle } \theta \text{ (written } \textit{tan } \theta) = \frac{\text{ordinate}}{\text{abscissa}} = \frac{AP}{OA} = \frac{y}{x}, \text{ when } x \neq 0.$$

$$\textit{cotangent of angle } \theta \text{ (written } \textit{cot } \theta) = \frac{\text{abscissa}}{\text{ordinate}} = \frac{OA}{AP} = \frac{x}{y}, \text{ when } y \neq 0.$$

$$\textit{secant of angle } \theta \text{ (written } \textit{sec } \theta) = \frac{\text{distance}}{\text{abscissa}} = \frac{OP}{OA} = \frac{r}{x}, \text{ when } x \neq 0.$$

$$\textit{cosecant of angle } \theta \text{ (written } \textit{csc } \theta) = \frac{\text{distance}}{\text{ordinate}} = \frac{OP}{AP} = \frac{r}{y}, \text{ when } y \neq 0.$$

There are three other functions in common use:

versed sine θ (written *vers* θ) $= 1 - \cos\theta$.

coversed sine θ (written *covers* θ) $= 1 - \sin\theta$.

$$\textit{haversine } \theta \text{ (written } \textit{hav } \theta) = \tfrac{1}{2} \textit{ versed sine } \theta = \frac{1 - \cos\theta}{2}.$$

These definitions of the functions hold for an angle of any size and include those given in Art. 22, which apply to any angle between 0° and 90°, as a special case.

31. Functions Are Constant for Any Angle.—It should be noted that the length OP in Fig. 12 was chosen arbitrarily but if a different value were given to OP the lengths OA and AP would change in the same ratio as OP, and hence the functions of θ would remain unaltered in value. Thus the trigonometric functions are constant for any angle.

32. Signs of the Functions.—The trigonometric ratios have been defined in terms of the distance, r, and x and y, the coordinates of a point at its extremity. The distance is always considered positive but as the radius vector revolves about the origin into different positions, the trigonometric functions vary in sign since they depend on x and y.

If $\theta = n \times 360° + a$, where $0° < a < 90°$, and n is a positive or negative integer or zero, r (see Fig. 12, a) is in the first quadrant and x and y are positive. Therefore, when the terminal side of the angle is in the first quadrant, all the functions are positive.

If $\theta = n \times 360° + a$ where $90° < a < 180°$, r falls in the second quadrant (see Fig. 12, b), y being directed upward is positive and x extending to the left is negative. Therefore, all the ratios involving x are negative. Hence, in the second quadrant the sine and cosecant are positive, the cosine, secant, tangent, and cotangent are negative.

If $\theta = n \times 360° + a$ where $180° < a < 270°$ (Fig. 12, c), r falls in the third quadrant and x and y are both negative. Therefore, only those ratios which involve both x and y, namely, the tangent and cotangent, are positive and the sine, cosine, secant, and cosecant are negative.

If $\theta = n \times 360° + a$ where $270° < a < 360°$ (Fig. 12, d), r falls in the fourth quadrant, x is positive but y is negative. Thus, in the fourth quadrant the cosine and secant are positive, the sine, cosecant, tangent, and cotangent are negative.

We may summarize these results for the six ratios in the following table:

II Sine and cosecant plus Others minus	I All plus
Tangent and cotangent plus Others minus III	Cosine and secant plus Others minus IV

It then follows from their definitions that the versed sine, coversed sine, and haversine are positive in all quadrants.

33. For a few angles it is possible to determine the values of the trigonometric functions from simple geometrical con-

structions. These angles and the values of the functions are of such common use that it is well to learn them. It is not recommended that the values of the functions be memorized as that is apt to lead to confusion, but rather that the method of construction be learned. The student should be able to make a *mental* picture of the angle and from it give the values of the functions.

Functions of 30°, 150°, 210°, 330°.

To find the functions of 30° lay off the angle as shown in Fig. 6.

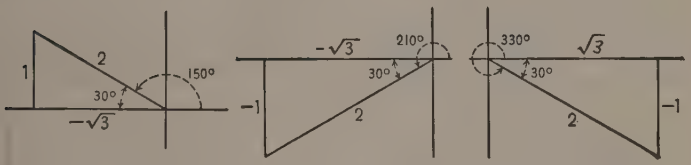


Fig. 13

The three figures (Fig. 13) serve for finding the functions of 150°, 210°, and 330°, respectively. Thus,

$\sin 150° = \frac{1}{2}$.	$\sin 210° = -\frac{1}{2}$.	$\sin 330° = -\frac{1}{2}$.
$\cos 150° = -\frac{\sqrt{3}}{2}$.	$\cos 210° = -\frac{\sqrt{3}}{2}$.	$\cos 330° = \frac{\sqrt{3}}{2}$.
$\tan 150° = -\frac{1}{3}\sqrt{3}$.	$\tan 210° = \frac{1}{3}\sqrt{3}$.	$\tan 330° = -\frac{1}{3}\sqrt{3}$.
$\cot 150° = -\sqrt{3}$.	$\cot 210° = \sqrt{3}$.	$\cot 330° = -\sqrt{3}$.
$\sec 150° = -\frac{2}{3}\sqrt{3}$.	$\sec 210° = -\frac{2}{3}\sqrt{3}$.	$\sec 330° = \frac{2}{3}\sqrt{3}$.
$\csc 150° = 2$.	$\csc 210° = -2$.	$\csc 330° = -2$.

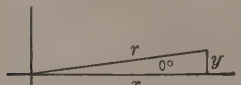
Functions of 45°, 135°, 225°, 315°.

In each of these cases the triangle is isosceles and for ease in the work we take each leg as of length 1, in which case the hypotenuse is of length $\sqrt{2}$.

Functions of 60°, 120°, 240°, 300°.

In each of these cases we have the same triangle as in the 30°, 150°, etc., case except that here the coördinates are interchanged and the ordinate is of length $\sqrt{3}$ and the abscissa of length 1.

34. Functions of 0°, 90°, 180°, 270°.—Since P is a point on the terminal side of the angle, such that the distance r is greater than zero, x and y cannot both be zero at the same time. When either x or y is zero, a function having that letter in the denominator is undefined. The cotangent and cosecant of 0° are undefined, but the trigonometric functions of 0° may be considered as limiting values. Take a very small angle as shown in Fig. 14. Then as the angle approaches 0°, y approaches 0 and x approaches r, and we have



$\lim x = r$
$\lim y = 0$

FIG. 14

$$\tan 0° = \lim_{\substack{y=0 \\ x=r}} \frac{y}{x} = 0.$$

$$\cot 0° = \lim_{\substack{y=0 \\ x=r}} \frac{x}{y} = \infty.$$

$$\sin 0° = \lim_{y=0} \frac{y}{r} = 0.$$

$$\sec 0° = \lim_{x=r} \frac{r}{x} = 1.$$

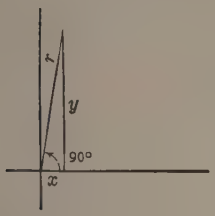
$$\cos 0° = \lim_{x=r} \frac{x}{r} = 1.$$

$$\csc 0° = \lim_{y=0} \frac{r}{y} = \infty.$$

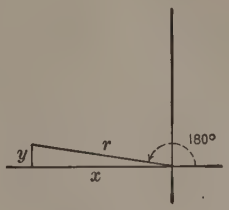
Strictly speaking, the limit of a fraction whose numerator is different from 0 and whose denominator approaches 0 does not exist. The value of the fraction increases indefinitely (becomes infinite) as the denominator approaches zero. However, we symbolize this as above and say

$$\lim_{X=0} \frac{A}{X} = \infty \text{ (infinity)}.$$

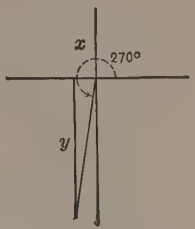
In a similar manner we find the functions of 90°, 180°, and 270°; the results are given below.



$\lim y = r$
$\lim x = 0$



$\lim x = -r$
$\lim y = 0$



$\lim x = 0$
$\lim y = -r$

FIG. 15

$$\sin 90° = \frac{r}{r} = 1. \qquad \sin 180° = \frac{0}{r} = 0. \qquad \sin 270° = \frac{-r}{r} = -1.$$

$$\cos 90° = \frac{0}{r} = 0. \qquad \cos 180° = \frac{-r}{r} = -1. \qquad \cos 270° = \frac{0}{r} = 0.$$

$$\tan 90° = \frac{r}{0} = \infty. \qquad \tan 180° = \frac{0}{-r} = 0. \qquad \tan 270° = \frac{-r}{0} = \infty.$$

$$\cot 90° = \frac{0}{r} = 0. \qquad \cot 180° = \frac{-r}{0} = \infty. \qquad \cot 270° = \frac{0}{-r} = 0.$$

$$\sec 90° = \frac{r}{0} = \infty. \qquad \sec 180° = \frac{r}{-r} = -1. \qquad \sec 270° = \frac{r}{0} = \infty.$$

$$\csc 90° = \frac{r}{r} = 1. \qquad \csc 180° = \frac{r}{0} = \infty. \qquad \csc 270° = \frac{r}{-r} = -1.$$

EXERCISE XV

1. Give functions of 0°, 30°, 45°, 60°, and 90°.
2. Give functions of 120°, 135°, 150°, and 180°.
3. Give functions of 210°, 225°, 240°, and 270°.
4. Give functions of 300°, 315°, 330°, and 360°.

Find the value of the following four expressions by inserting numerical values of the trigonometric functions:

5. $4 \cos 60° + 3 \tan 45° + 2 \sin 30°$.

6. $4\sqrt{3} \cos 150° + 3 \sin 90° \cdot \tan 225° - 6 \sin 330° + \cos 270°$.

7. $4 \sin 90° \cdot \cos 300° \cdot \cos 0° + \frac{1}{\sqrt{3}} \cdot \tan 240°$.

8. $\sin 270° \cdot \cos 180° \cdot \tan 45° \cdot \cot 135°$.

9. Does the $\sin 60° = 2 \sin 30°$?
$\tan 90° = 2 \tan 45°$?
$\cos 180° = 3 \cos 60°$?
What can be said of the truth of the following?
$\sin 2\theta = 2 \sin \theta$; $\sin 3\theta = 3 \sin \theta$,
$\tan 2\theta = 2 \tan \theta$; etc.

10. Does an angle exist whose sine is 2 or 3 or -4? Whose cosine is 2, 3, etc.?

11. Does an angle exist whose secant is $\frac{1}{2}$ or $-\frac{1}{3}$? Whose cosecant is $\frac{1}{2}$?

12. Show that $\cos 90° = \cos 60° \cos 30° - \sin 60° \sin 30°$.

13. Show that $\tan 210° = \dfrac{\tan 240° - \tan 30°}{1 + \tan 240° \tan 30°}$.

14. Show that $\sin 60° = \sin 90° \cos 30° - \cos 90° \sin 30°$.

15. Show that $\cot 120° = \dfrac{\cos 150° \cos 30° + \sin 150° \sin 30°}{\sin 150° \cos 30° - \cos 150° \sin 30°}$.

16. Make a table of the haversines of 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°, 300°, 315°, and 330°.

35. Given One Function of an Angle; to Find the Remaining Functions by Construction.

Example 1.—Given $\cos \theta = \frac{4}{5}$, construct the angle and give the values of the other functions.

By definition $\cos\theta=\frac{x}{r}=\frac{4}{5}$. Since r is always positive the abscissa must also be positive. Hence the abscissa must be measured to the right of the origin. Since we are concerned only with the ratio we may take $x=4$ and $r=5$. In Fig. 16 using any unit lay off $x=4$ and draw a line PQ through this point parallel to OY. With O as a center and a radius of 5 units describe an arc cutting PQ at A and B. Draw OA and OB. Then either $\angle XOA$ or $\angle XOB$, each of which has 4/5 for its cosine, is the angle θ. Since $OA=5$ and $OC=4$ then $CA=3$; $CB=-3$. Then we have $\sin\theta=\pm 3/5$, $\cos\theta=4/5$, $\tan\theta=\pm 3/4$, $\cot\theta=\pm 4/3$, $\sec\theta=5/4$, $\csc\theta=\pm 5/3$.

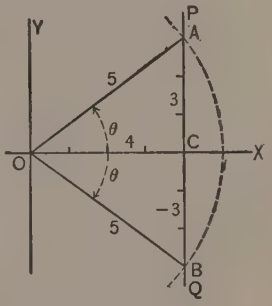


Fig. 16

Example 2.—Given $\cot\theta=-\frac{8}{15}$, construct the angle and give the values of the other functions.

By definition $\cot\theta=\frac{x}{y}=-\frac{8}{15}$. Since y and x can both be either positive or negative we may have either,

$$\frac{x}{y}=\frac{-8}{15} \text{ or } \frac{8}{-15}.$$

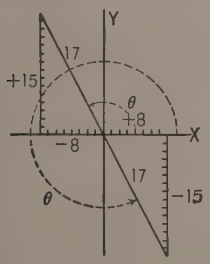


Fig. 17

In Fig. 17 lay off 8 units to the left for $x=-8$ and 15 units up for $y=15$, in which case we get θ in the second quadrant. Lay off 8 units to the right for $x=8$ and 15 units down for $y=-15$ and we get θ in the fourth quadrant. In either case $r=17$.

Thus,

$$\sin\theta = \pm\tfrac{15}{17}. \qquad \cot\theta = -\tfrac{8}{15}.$$
$$\cos\theta = \mp\tfrac{8}{17}. \qquad \sec\theta = \mp\tfrac{17}{8}.$$
$$\tan\theta = -\tfrac{15}{8}. \qquad \csc\theta = \pm\tfrac{17}{15}.$$

EXERCISE XVI

In Exercises 1 to 6 give the values of the remaining functions.

1. $\sin\theta = \frac{1}{2}$.

2. $\tan\theta = 2$.

3. $\sec\theta = -\frac{5}{2}$.

4. $\cot\theta = \frac{1}{3}$.

5. $\cos\theta = -\frac{1}{2}\sqrt{3}$.

6. $\sec\theta = 3$.

7. Make a chart showing the signs of the six functions in all four quadrants.

8. Find the value of

$$\frac{\sin\theta + \cos\theta - \tan\theta}{\sec\theta + \csc\theta - \tan\theta}$$

when $\cot\theta = \frac{4}{3}$ and θ is in the third quadrant.

9. Evaluate **8** for θ in the first quadrant.

10. Determine the value or values of θ (less than 360°) for each of the following:

(*a*) $\tan\theta = 1$, θ in 3d quadrant.
(*b*) $\sin\theta = -\frac{1}{2}$, θ in 4th quadrant.
(*c*) $\sec\theta = 2$, θ in 1st quadrant.
(*d*) $\sin\theta = 1$.
(*e*) $\cos\theta = 0$.
(*f*) $\cot\theta = -1$, $90° < \theta < 180°$.
(*g*) $\cot\theta = \sqrt{3}$.
(*h*) $\csc\theta = \sqrt{2}$.
(*i*) $\tan\theta = -\frac{1}{3}\sqrt{3}$.
(*j*) $\sec\theta = 1$.
(*k*) $\sec\theta = -2$.
(*l*) $\sin\theta = \dfrac{\sqrt{3}}{2}$.

36. The Six Ratios Are Not Independent.—For example, from the definition it follows that

$$\sin\theta = \frac{1}{\csc\theta} \quad \text{or} \quad \sin\theta \cdot \csc\theta = 1. \tag{1}$$

$$\cos\theta=\frac{1}{\sec\theta} \quad \text{or} \quad \cos\theta\cdot\sec\theta=1. \tag{2}$$

$$\tan\theta=\frac{1}{\cot\theta} \quad \text{or} \quad \tan\theta\cdot\cot\theta=1. \tag{3}$$

These relations are known as the reciprocal relations between the functions.

37. Exponents of the Functions.—Whenever the trigonometric functions are to be affected by exponents the notation in general is $\sin^2\theta$ for $(\sin\theta)^2$, $\cos^3\theta$ for $(\cos\theta)^3$, $\csc^n\theta$ for $(\csc\theta)^n$, $\tan^{-3}\theta$ for $(\tan\theta)^{-3}$, etc. The only ambiguity which may arise is in the case of the exponent -1; this has a particular significance which will be discussed in a later section. To avoid confusion in this case it is best to write it as $(\sin\theta)^{-1}$, $(\cos\theta)^{-1}$, $(\sec\theta)^{-1}$, etc.

38. Formulae.—In any of the four quadrants (see Fig. 12)

$$x^2+y^2=r^2; \tag{a}$$

dividing both sides of this equation by r^2, we have

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}=\frac{r^2}{r^2}=1,$$

which can be written,

$$\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2=1;$$

but by the definitions of the trigonometric functions

$$\frac{y}{r}=\sin\theta, \quad \frac{x}{r}=\cos\theta.$$

Therefore, $$\sin^2\theta+\cos^2\theta=1. \tag{4}$$

This formula is of fundamental importance and should be learned so that it will be recognized in its various forms as $\sin^2\theta=1-\cos^2\theta$; $\cos^2\theta=1-\sin^2\theta$; $\sin\theta=\pm\sqrt{1-\cos^2\theta}$, etc.

When $x \neq 0$ we can divide both sides of (a) by x^2 and we have

$$1+\left(\frac{y}{x}\right)^2=\left(\frac{r}{x}\right)^2,$$

or
$$\mathbf{1+\tan^2\theta=\sec^2\theta;} \tag{5}$$

when $y \neq 0$,

$$\left(\frac{x}{y}\right)^2+1=\left(\frac{r}{y}\right)^2,$$

or
$$\mathbf{\cot^2\theta+1=\csc^2\theta} \tag{6}$$

From the definition of the tangent we have

$$\tan\theta=\frac{y}{x}=\frac{y/r}{x/r}=\frac{\sin\theta}{\cos\theta}, \text{ when } x \neq 0; \tag{7}$$

and for the cotangent,

$$\cot\theta=\frac{x}{y}=\frac{x/r}{y/r}=\frac{\cos\theta}{\sin\theta}, \text{ when } y \neq 0. \tag{8}$$

39. All Functions in Terms of One Function.—The formulae expressing all the functions in terms of one function can very easily be obtained by using a right triangle, as shown by the following example.

Example.—Express all the functions of θ in terms of $\cos\theta$. Solution.—$\cos\theta = x/r$. Therefore, in the right triangle of Fig. 18, take $r=1$. Then $x=\cos\theta$ and $y=\sqrt{1-\cos^2\theta}$. (The sign before $\sqrt{1-\cos^2\theta}$ depends on the quadrant of θ.)

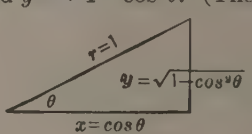


FIG. 18

$\sin\theta = y/r = \sqrt{1-\cos^2\theta}$.

$\cos\theta = x/r = \cos\theta$.

$$\tan\theta=y/x=\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}. \qquad \sec\theta=r/x=\frac{1}{\cos\theta}.$$

$$\cot\theta=x/y=\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}. \qquad \csc\theta=r/y=\frac{1}{\sqrt{1-\cos^2\theta}}.$$

EXERCISE XVII

1. Express all the functions in terms of sin θ; of the cot θ; of the sec θ.

2. Express all the functions in terms of tan θ; of the csc θ.

3. If tan $\alpha = \frac{2}{4}\frac{9}{6}\frac{1}{3}$ find the remaining functions of α to 4 decimals.

4. If $\cos\theta = \dfrac{a^2-b^2}{a^2+b^2}$ find the remaining functions of θ.

5. If cot $\alpha = 0.387$ find correct to 3 decimals the value of $\dfrac{5\sin\alpha - 3\cos\alpha}{\sin\alpha + 2\cos\alpha}$.

40. In algebra we are familiar with two kinds of expressions of equality, i.e., those which are true for all values of the unknown and those which are true for a restricted number of values. For example:

$$x^2-5x+6=(x-2)(x-3)$$

is an identical equation because both sides of the equality have the same value for all values of x; whereas

$$x^2-5x+6=0$$

is true only for $x=2$ and $x=3$.

Both of these types of equalities arise in equations which involve trigonometric functions. The formulae (1) to (8) are identical equations; that is, they are true for all values of the angle for which the functions involved are defined.

Consider the equation

(*a*) $$\cos A\ (\cot A+\tan A)=\csc A,$$

by (7) and (8) we have

$$\cos A\left(\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)=\csc A.$$

Reducing to a common denominator

$$\cos A\left(\frac{\cos^2 A+\sin^2 A}{\sin A\cos A}\right)=\csc A.$$

Cancelling the $\cos A$ and replacing $\cos^2 A+\sin^2 A$ by 1, we have

$$\frac{1}{\sin A}=\csc A$$

which by (1) is true. Hence the identity (a) is true since the first expression can be transformed into the second.

(b) $\sin^4 A+\cos^4 A=1-2\sin^2 A\cos^2 A.$

$\sin^4 A+\cos^4 A=$

$\sin^2 A\cdot\sin^2 A+\cos^2 A\cdot\cos^2 A=$

$\sin^2 A\,(1-\cos^2 A)+\cos^2 A\,(1-\sin^2 A)=$ (by 4)

$\sin^2 A-\sin^2 A\cos^2 A+\cos^2 A-\sin^2 A\cos^2 A=$

$\sin^2 A+\cos^2 A-2\sin^2 A\cos^2 A=$

$1-2\sin^2 A\cos^2 A.$ (by 4)

(c) $$\frac{\sec A-\csc A}{\sec A+\csc A}=\frac{\tan A-1}{\tan A+1}.$$

Transforming both sides,

$$\frac{\dfrac{1}{\cos A}-\dfrac{1}{\sin A}}{\dfrac{1}{\cos A}+\dfrac{1}{\sin A}}=\frac{\dfrac{\sin A}{\cos A}-1}{\dfrac{\sin A}{\cos A}+1},$$ by (1), (2), (7), and (8)

or

$$\frac{\dfrac{\sin A-\cos A}{\sin A\cos A}}{\dfrac{\sin A+\cos A}{\sin A\cos A}}=\frac{\sin A-\cos A}{\sin A+\cos A},$$

or

$$\frac{\sin A-\cos A}{\sin A+\cos A}=\frac{\sin A-\cos A}{\sin A+\cos A}.$$

In verifying identities such as (*a*), (*b*), and (*c*) any algebraic process is permissible. It must be noted, however, that in (*a*) and (*c*) the real proof is the reverse of what is given. For example, in (*a*) we have proved that if $\cos A\ (\cot A+\tan A) = \csc A$ then $\frac{1}{\sin A}=\csc A$; the proof that (*a*) is an identity, however, starts with $\frac{1}{\sin A}=\csc A$ and goes backward step for step to (*a*). Hence the test of the validity of the proof lies in the reversibility, step for step, of the process used.

EXERCISE XVIII

Verify the following identities:

1. $(\sin x+\cos x)\ (\tan x+\cot x)=\sec x+\csc x.$

2. $\cos A \csc A \tan A=1.$

3. $\tan\theta \sin\theta+\cos\theta=\sec\theta.$

4. $\cos^2 A\ (1+\tan^2 A)=1.$

5. $\frac{\sin A}{\csc A}+\frac{\cos A}{\sec A}=1.$

6. $\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}.$

7. $\frac{1-\tan x}{1+\tan x}=\frac{\cot x-1}{\cot x+1}.$

8. $(\sec^2 A-1)\ (\csc^2 A-1)=1.$

9. $\sin^6 A+\cos^6 A=1-3\sin^2 A\cos^2 A.$

10. $\sin^2 x-\cos^2 x=2\sin^2 x-1=1-2\cos^2 x.$

11. $\frac{1+\cot^2 A}{1+\tan^2 A}=\cot^2 A.$

12. $\dfrac{1+\sin\theta}{1-\sin\theta}-\dfrac{1-\sin\theta}{1+\sin\theta}=4\tan\theta\sec\theta.$

13. $\sqrt{\dfrac{1-\sin x}{1+\sin x}}=\sec x-\tan x.$

14. $\sec A-\cos A=\sin A\tan A.$

15. $\dfrac{\sin A}{1+\cos A}+\dfrac{1+\cos A}{\sin A}=2\csc A.$

16. $\cos^4 A-\sin^4 A=\cos^2 A-\sin^2 A.$

17. $\sqrt{\dfrac{\csc\theta-\cot\theta}{\csc\theta+\cot\theta}}=\csc\theta-\cot\theta.$

18. $(\csc\theta-\cot\theta)^2=\dfrac{1-\cos\theta}{1+\cos\theta}.$

19. $\cot y+\dfrac{\sin y}{1+\cos y}=\csc y.$

20. $\dfrac{\csc y}{\csc y-1}+\dfrac{\csc y}{\csc y+1}=2\sec^2 y.$

21. $\dfrac{\csc\theta}{\tan\theta+\cot\theta}=\cos\theta.$

22. $\sec^2\theta-\sec^2\theta\sin^2\theta=1.$

23. $\tan x+\cot x=\sec x\csc x.$

24. $(\tan\theta+\cot\theta)^2=\sec^2\theta+\csc^2\theta.$

25. $\sec^2 x+\tan^2 y=\sec^2 y+\tan^2 x.$

26. $(1-\sin^2 A)\csc^2 A=\cot^2 A.$

27. $\tan^2 A+\cot^2 A=\sec^2 A\csc^2 A\,(1-2\sin^2 A\cos^2 A).$

28. $\dfrac{\sec\theta}{\cos\theta}-\dfrac{\tan\theta}{\cot\theta}=1.$

29. $\sec A+\tan A=(\sec A-\tan A)^{-1},$

30. $1-\cot^4\theta=2\csc^2\theta-\csc^4\theta,$

CHAPTER V

THE RIGHT TRIANGLE

41. Use of Tables of Logarithms of Trigonometric Functions for Angles Equal to or Less Than 90°.—Suppose we desire to find the value of $a = c \sin A$; where $c = 24.895$ and $A = 44°10'36''$. $N \sin 44°10'36'' = 0.69688$ and we have

$$a = 24.895 \times 0.69688$$

Rather than actually multiply we would find a by logarithms as follows:—

24.895	log 1.39611
0.69688	log 9.84316−10
$a = 17.349$	log 1.23927

In other words, we would still make the computation by means of logarithms. The logarithm of $N \sin 44°10'36''$ is called the Logarithmic Sine $44°10'36''$. Obviously, since a great deal of computation is performed by means of logarithms it would be a distinct advantage to have the logarithmic sines tabulated so that it would not always be necessary to first find the Natural sine of the angle.

Should we take the logarithms of all the numbers in the Natural Sine table and tabulate them we would have a table of Logarithmic Sines. If the table of Natural Sines gives the values for every minute from 0° to 90° then the logarithmic table so computed would give the logarithmic sines for every minute from 0° to 90°. The same discussion applies exactly to the other trigonometric functions. Should we carry out the work we could complete a table for all six of the trigonometric ratios.

In short, then, the table of Logarithms of Trigonometric Functions is merely a tabulation of the logarithms of the actual

values of the ratios for the values of the angle as tabulated for every minute from 0° to 90°.

As a matter of practice, the tables are not actually constructed in this way as there are other methods which are quicker and more accurate.

42. Suggestions will now be offered with regard to the use of five-place tables for angles less than 90°. It has been pointed out that the value of the sine is zero at 0°, and increases to unity at 90°; correspondingly the cosine decreases from unity to zero. In other words the sine and cosine vary from 0 to 1. Hence, the Table of Log Sines and Cosines will be made up almost wholly of logarithms with negative characteristics. The single exception will be the logarithm of 1 which is zero; this occurs for the log sin 90° and the log cos 0°.

The tables give the positive part of the characteristic but not the negative. Thus the table gives the log sin 36° as 9.76922 but the -10 must be supplied or at least understood; the correct value is log sin $36° = 9.76922 - 10$.

If the number of degrees is at the top of the page, use the log column having the name of the function at the top. If the number of degrees is at the bottom, use the log column having the name of the function at the bottom.

Table 44, Bowditch, gives the values of all six Logarithmic Functions to five decimals, for angles given to minutes. If the angle is given to seconds it is necessary to interpolate in the usual way. Thus, if n represents the number of seconds, D the difference between the logarithmic function of the next lesser number of minutes and the logarithmic function of the next greater number of minutes, and d the difference between the logarithmic function of the next lesser number of minutes and that of the required angle,

$$d = n/60 \times D.$$

Table 44 is so arranged that a method of interpolation by inspection is possible except within 5° of the ends of the table.

There are three difference (Diff.) columns, one for the sine and cosine, one for the tangent and cotangent, one for the secant and cosecant. To use this method look up n in the left-hand minute column (headed M) and read d on a horizontal line with n from the proper Diff. column.

Care must be taken in interpolation whether by means of the difference tables or otherwise, to add the correction only when it should be added; i.e., when the function is increasing as the angle increases. When the function is decreasing as the angle increases, the correction must be subtracted.

When the log function is known and the angle has to be found from the table the operations are reversed just as was explained previously.

When interpolating for seconds in finding the angle, it often happens that in the difference tables the *difference* between the given log and that of the log of the exact minute is repeated several times in the difference column. In this case take the number of seconds midway of the range.

EXERCISE XIX

1. Verify the following:

$\log \sin 37°18'00'' = 9.78246 - 10.$
$\log \sec 23°51'00'' = 0.03877.$
$\log \cot 29°23'00'' = 0.24942.$
$\log \csc 73°37'00'' = 0.01800.$
$\log \cos 69°49'00'' = 9.53785 - 10.$
$\log \sec 36°18'00'' = 0.09370.$
$\log \csc 22°53'00'' = 0.41021.$
$\log \sin 32°17'39'' = 9.72776 - 10.$
$\log \cos 17°37'15'' = 9.97913 - 10.$
$\log \tan 28°19'53'' = 9.73171 - 10.$
$\log \cot 15°27'35'' = 0.55820.$
$\log \csc 48°17'52'' = 0.12690.$

2. Verify the following angles found from the given log functions:

log sin $\theta = 9.81237 - 10$,	$\theta = 40°28'48''$.
log tan $x = 0.37232$,	$x = 67°00'28''$.
log sec $\theta = 0.02179$,	$\theta = 18°00'00''$.
log csc $y = 0.51005$,	$y = 17°59'54''$.
log cos $x = 9.42902 - 10$,	$x = 74°25'20''$.

3. The horizontal range of a projectile is given by the formula

$$X = \frac{V_0^2 \sin 2\alpha}{g},$$

where V_0 = initial velocity, α = angle of elevation, and g = acceleration of gravity. Find X when $V_0 = 2500$ ft. per sec., $\alpha = 16°28'40''$, $g = 32.16$.

4. The height of a projectile after t seconds is given by the formula: $h = V_0 t \sin \alpha - \frac{1}{2} g t^2$, where V_0 = initial velocity, α = angle of elevation, g = acceleration of gravity. Find h when $t = 3.5$ sec., $V_0 = 1800$ ft. per sec., $\alpha = 9°10'22''$, and $g = 32.16$ ft. per sec. per sec.

5. The force to prevent slipping on an inclined plane is given by

$$F = W \frac{\sin (\alpha - \phi)}{\cos (\beta + \phi)},$$

where W = weight of body, α = inclination of plane, β = angle of force F, ϕ = angle of repose. Find F when $W = 368$ lbs., $\alpha = 45°$, $\beta = 28°$, $\phi = 18°26'06''$.

6. The acceleration of gravity is given by the formula $g = 32.088\ (1 + k \sin^2 L)$, where L = Latitude, k = constant. If $g = 32.209$ in Latitude $58°27'$, find k.

7. The force P which will prevent a wedge being forced upward by two horizontal loads each W is given by the formula

$P=\dfrac{2W\cos\phi\sin(\alpha-\phi)}{\cos(\alpha-2\phi)}$, where 2α is the angle of the wedge and ϕ is the angle of friction. Find P when $W=1000$ lbs., $\tan\phi=0.1$, and $\tan\alpha=1/8$.

8. The moment of the frictional force on a conical pivot is given by the formula $M=P/2r\tan\phi\csc\alpha$, where $P=$the axial load, $r=$radius of pivot, $\phi=$angle of friction, $2\alpha=$vertical angle of cone. Find M when $P=1000$ lbs., $r=0.5$ in., $2\alpha=30°$, $\phi=6°50'$.

43. Solution of Right Triangle.—The right triangle can be solved (i.e., the unknown parts can be found) if in addition to the right angle, one side and any other part are known. That is, given the fact that the triangle is a right triangle, we can solve it completely if we know two sides, or one side and an acute angle. By a side here we mean either a leg or the hypotenuse. Applying the definitions of the functions to the right triangle of Fig. 19 we obtain

$$\sin A=\frac{a}{c}=\cos B.$$

$$\cos A=\frac{b}{c}=\sin B.$$

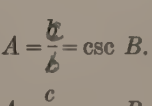


FIG. 19

$$\tan A=\frac{a}{b}=\cot B.$$

$$\sec A=\frac{c}{b}=\csc B.$$

$$\cot A=\frac{b}{a}=\tan B.$$

$$\csc A=\frac{c}{a}=\sec B.$$

These in connection with the two formulae $a^2+b^2=c^2$ and $A+B=90°$, which are obtained from geometry, form the basis of the solution.[1]

In making solutions it is recommended that a figure be drawn with fair accuracy by means of a ruler, compasses, and protractor in order to have an approximate check on the results.

[1] It is conventional to letter the right triangle as in Fig. 19.

The work should be laid out carefully in accordance with the conventions for computation. A check may be put on the work by employing other formulae than the ones selected for the solution.

Example 1.—Given the hypotenuse and an acute angle; to solve the triangle.

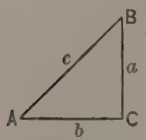


FIG. 20

Given c and A (Fig. 20):
To find B; $B=90°-A$;

To find a; $\sin A=\frac{a}{c}$; $a=c \sin A$.

To find b; $\cos A=\frac{b}{c}$; $b=c \cos A$.

Check: $\tan B=\frac{b}{a}$.

$c=24.895$, $A=44°10'38''$.
Therefore, $B=45°49'22''$.

Solution by Logarithms

$A=44°10'38''$	sin 9.84316	cos 9.85563
$c=24.895$	log 1.39611	log 1.39611
$b=17.854$		log 1.25174
$a=17.349$	log 1.23927	colog 8.76073
$B=45°49'22''$		tan 0.01247
Check→		←*Check*

Example 2.—Given, $a=3.2914$, $b=5.7842$; $\tan A=\frac{a}{b}$, $c=b \sec A=a \csc A$.

$a=3.2914$	log 0.51738		log 0.51738
$b=5.7842$	colog 9.23776	log 0.76224	
$A=29°38'28''$	tan 9.75514	sec 0.06091	csc 0.30578
$c=6.6550$		log 0.82315	log 0.82316
		→*Check*←	

$B=90°-A=60°21'32''$.

This is a satisfactory check on the logarithm of c and the value of c obtained from either logarithm would be consistently accurate; the apparent discrepancy is inherent in the table. (See Art. 9.) In finding A there was a choice of $28''$ or $29''$.

Limitations in the accuracy of the five-place tables of Logarithms of Numbers have been pointed out; at this time it is well to note a corresponding limitation in the accuracy of the tables of Log Sines, Tangents, and Secants. The degree of accuracy depends to some extent upon the function and the place in the table. For example: between 44° and 45° the tabular difference for the log sine is 13, and an angle in this range found from its log sine is only determined to an interval of $\frac{60}{13}$ or 5 secs. This can readily be verified by observing the column of differences wherein each difference occurs four or five times. In this same range the tabular difference for the tangent is 25 and hence it is accurate to 3 secs. On the other hand the sine in the range from 6° to 7° has a tabular difference of 111 and hence is accurate to $\frac{60}{111}$ or nearly $\frac{1}{2}''$. If greater accuracy than has been pointed out is desired a table should be used with sufficient places to insure it.

Further, in practical problems the results should be consistent with the given or observed data. If the given data are accurate to five places there is no object in computing results to more places than this. If the given data are only accurate to minutes of angle the results should be given only to minutes.

However, for practice in the use of the five-place tables, results will be carried to five significant figures for numbers and to seconds for angles.

The area of a triangle may be expressed in several ways and is usually notated as K. The following are expressions for the area of a right triangle.

$$K = \tfrac{1}{2}\, a \cdot b = \tfrac{1}{2}\, a \cdot c \cdot \cos A = \tfrac{1}{2}\, b \cdot c \cdot \sin A.$$
$$K = \tfrac{1}{2}\, b^2 \cdot \tan A = \tfrac{1}{2}\, a^2 \cdot \tan B, \text{ etc.}$$

44. Applications.—In some of the succeeding examples the following definitions will be necessary:

A vertical line is one determined by a plumb-line.

A vertical plane is a plane passing through a vertical line.

A horizontal plane is a plane perpendicular to a vertical line.

For small distances any liquid at rest may be considered as having a horizontal surface.

The angle which a line from the eye of an observer to an object makes with a horizontal line in the same vertical plane is called an *angle of elevation* or of *depression*, according as the object is above or below the observer's eye. (See Fig. 21.)

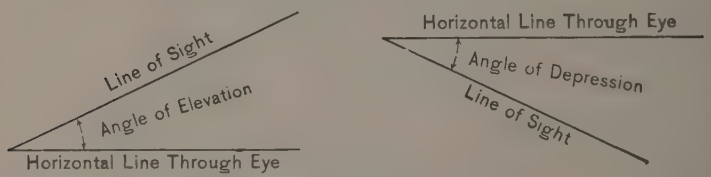


FIG. 21

45. Applications in Navigation.—In the applications which we shall give in this chapter we shall assume that the distances considered are sufficiently small so that the curvature of the earth may be neglected.

Directions on the surface of the sea are usually given by direction as located on the mariner's compass. Figure 22 shows the 32 divisions, $11\frac{1}{4}°$ apart, called points, on the periphery of a compass card. These are further divided into half-points and quarter-points, and sometimes eighth-points.

FIG. 22

The directions on this compass are usually spoken of as bearings.

All compasses in the naval service are now graduated in degrees beginning with 0° at the north point and proceeding through the east, south and west back to 360° at the north point. The positive direction here is clockwise which is contrary to the positive direction of angles as used generally in trigonometry.

An illustration of the modern Navy compass is given in Fig. 23.

FIG. 23

In considering a ship's position at sea with reference to any other place, either one that the vessel has left or one toward which it is bound, five terms are involved—the Course, the Distance, the Difference of Latitude, the Difference of Longitude, and the Departure.

When the only quantities involved are the course, distance, difference of latitude and departure, the process is denominated Plane Sailing. In this method the earth is regarded as a plane.

Plane sailing furnishes no solution where difference of longitude is concerned.

In plane sailing the relations between the elements of the rhumb track[1] joining any two points may be considered from the plane right triangle formed by the meridian of the place left, the parallel of the place left, the parallel of the place arrived at, and the rhumb line. In Fig. 24; T is the point of departure;

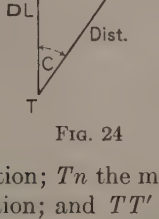


Fig. 24

T' the point of destination; Tn the meridian of departure; $T'n$ the parallel of destination; and TT' the rhumb line between the points. Let C represent the course, $T'Tn$ (the angle between the meridian and the rhumb line); Dist., the distance TT'; DL, the difference of latitude Tn; and Dep., the departure $T'n$ (the linear distance measured on a parallel of latitude between the two meridians).

Illustration.— A ship sails SW by W 244 miles. Required the difference of latitude and the departure made good.

	$\cos C = \frac{Tn}{\text{Dist}}$	$\sin C = \frac{nT'}{\text{Dist}}$
Dist = 244	log 2.38739	log 2.38739
$C = 56° 15'$	cos 9.74474	sin 9.91985
$DL = 135.6$ miles	log 2.13213	
Dep = 202.9 miles.		log 2.30724

[1] The curved line which joins any two places on the earth's surface cutting all the meridians at the same angle; in the case of plane sailing the Rhumb Line or Loxodromic curve is a straight line.

A Nautical Mile is a minute of arc on the equator and is approximately 6080 ft. (See Problem 2, Ex. XI.)

A knot is a unit of speed; 10 knots means 10 nautical miles per hour.

Example.—An obstacle separates the points A and B, but B is known to be 5170 feet distant and to bear $N\ 61°27'E$ from A. To locate the point B a line AC, 2000 feet long and bearing $N\ 37°42'E$, then a line CB is run. Find the bearing and distance of B from C.

In Fig. 25:

$BD = 5170 \sin 61°27'$,

$DF = EC = 2000 \sin 37°42'$,

$DA = 5170 \cos 61°27'$,

$EA = 2000 \cos 37°42'$,

$FB = BD - DF$,

$FC = DE = DA - EA$,

$\tan \alpha = \dfrac{FB}{FC}$, $CB = FB \csc \alpha$.

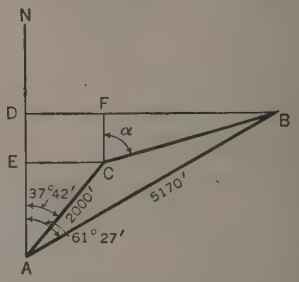


FIG. 25

5170	log 3.71349	log 3.71349
61°27′	sin 9.94369	cos 9.67936
$BD = 4541.3$	log 3.65718	
$DA = 2470.9$		log 3.39285
2000	log 3.30103	log 3.30103
37°42′	sin 9.78642	cos 9.89830
$DF = 1223.0$	log 3.08745	
$EA = 1582.4$		log 3.19933
$FB = 3318.3$	log 3.52092	log 3.52092
$FC = 888.5$	colog 7.05134	
$\alpha = N\ 75°\ 00'\ 36''E$	tan 0.57226	csc 0.01504
$BC = 3435.2$ feet		log 3.53596

EXERCISE XX

Solve the following ten right triangles:

1. $a = 396.25$, $b = 645.32$.
2. $c = 98.245$, $a = 64.532$.
3. $B = 27°9'14''$, $a = 35.421$.
4. $c = 672.341$, $A = 35°16'25''$.
5. $c = 42567$, $B = 87°49'10''$.
6. $c = 672.34$, $b = 548.90$.
7. $A = 88°59'$, $b = 2.23487$.
8. $A = 25°18'48''$, $a = 0.085623$.
9. $a = 8.6789$, $b = 2.4639$.
10. $a = 184.29$, $A = 28°47'45''$.

11. If the chord of a circle is one-half the radius, what is the size of the angle subtended at the center?

12. If a tower casts a shadow which is two-thirds its own length, what is the angle of elevation of the sun?

13. At a distance of 62.4 feet from the base of a tower the angle of elevation of its top is observed to be $62°27'12''$. Find the height of the tower.

14. If the radius of a circle is 12 feet, what is the size of the arc included between two parallel chords, on the same side of the center, whose distances from the center are 6 and 8 feet respectively?

15. From the top of a cliff 587 feet above sea level, the angles of depression of two boats, in line with the observer, are $18°10'27''$ and $28°45'32''$ respectively. Find the distance between the boats.

16. A compass is 6 inches in diameter. What is the length in inches on the circumference from N to ENE? From N to SW by W? From N to SSE?

17. What does the bearing SSW correspond to on a modern compass? the bearing E by S?

18. The 30 foot diameter of a lighthouse subtends an angle equal to 1° at the compass. What is the distance between the two?

19. A new road is to be constructed connecting the two towns A and B. B is 32 miles northeast of A. Due to the nature of the terrain it is necessary to construct the first 15 miles of the road from A in the direction $N\ 24°25'\ E$; assuming the remainder of the road to be straight, what must be its length and direction?

20. A ship sails NE by E at 17 knots. Find the rate at which it is moving north; also the rate at which it is moving east.

21. A is known to have an altitude of 7856 ft.; B to have an altitude of 15724 ft. The angle of elevation of B at A is 32°17′. Compute the distance from A to B.

22. A ladder 46 ft. long rests against a building and makes an angle of 69°47′ with the horizontal. Find the distance it reaches up the building.

23. If the foot of the ladder in Example 22 were moved 10 feet further from the wall, how far down the wall will the top of the ladder fall?

24. A circus tent is built in the form of a cone surmounting a cylinder. The altitude of the cylinder is 20 feet and that of the cone is 30 feet. The vertical angle of the cone is 125°. Find the number of square yards of canvas in the tent.

25. A hillside makes an angle of 23°10′ with the horizontal. A tunnel 211.5 feet long is bored in the hill; the tunnel slopes downward and makes an angle of 8°15′ with the horizontal. How far below the surface of the hill is a point 87 feet down the tunnel?

26. At a point 30 feet above the surface of a lake, the angle of elevation of the top of a tree standing on the shore is 37°29′; while the angle of depression of its image in the water is 61°18′. Find the height of the tree and its horizontal distance from the observer.

27. A ship sailing west at 17 knots observes a lighthouse at 1 o'clock to be N 65°43′ W. At 1:47 the lighthouse is N 53°38′ W. Determine at what time the ship is nearest the lighthouse. What is the distance between them at that time?

28. A ship sails a course 19° for 201 miles. Find the difference of latitude and the departure.

29. A vessel is bound to a port which is 187 miles to the north and 38 miles to the west of her position. What course must she steer and how long will it take her if her average speed is 12 knots?

30. As a result of a day's run a vessel changes latitude 233 miles to the south and makes a departure of 165 miles to the east. What was the course steered and distance made good? What was her average speed?

CHAPTER VI

FUNCTIONS OF RELATED ANGLES AND VARIATION OF THE FUNCTIONS

46. In this chapter it will be shown that the functions of certain angles are equal to functions of other related angles. For example, in Art. 33 it is shown that the functions of 150°, 210°, 330° are equal numerically to the same functions of 30°. Also the functions of 120°, 240°, 300° are equal numerically to the cofunctions of 30°. When angles are related in certain ways we can formulate certain relations between their functions.

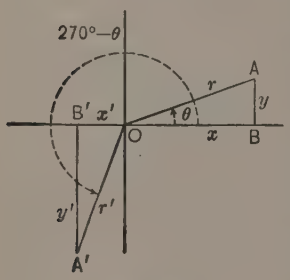


FIG. 26

47. Functions of θ and Functions of (270° − θ).—As a further illustration consider the angles θ and $270° - \theta$. The figure is drawn for values of θ in the first quadrant (Fig. 26).

$$\triangle OA'B' = \triangle OAB.$$

Therefore, $r' = r$, $y' = -x$, $x' = -y$, being homologous sides of equal triangles. These equations hold both for magnitude and direction.

Thus,
$$\sin (270° - \theta) = y'/r' = -x/r = -\cos \theta.$$
$$\cos (270° - \theta) = x'/r' = -y/r = -\sin \theta.$$
$$\tan (270° - \theta) = y'/x' = x/y = \cot \theta.$$
$$\cot (270° - \theta) = x'/y' = y/x = \tan \theta.$$
$$\sec (270° - \theta) = r'/x' = -r/y = -\csc \theta.$$
$$\csc (270° - \theta) = r'/y' = -r/x = -\sec \theta.$$

If figures are constructed for a θ in each of the other quadrants it can be shown that these same relations are true, and, therefore, these relations hold for every value of θ.

48. Functions of θ and Functions of $(-\theta)$.—Let the student draw a figure and prove geometrically:

$$\sin(-\theta) = y'/r' = -y/r = -\sin\theta.$$
$$\cos(-\theta) = x'/r' = x/r = \cos\theta.$$
$$\tan(-\theta) = y'/x' = -y/x = -\tan\theta.$$
$$\cot(-\theta) = x'/y' = x/-y = -\cot\theta.$$
$$\sec(-\theta) = r'/x' = r/x = \sec\theta.$$
$$\csc(-\theta) = r'/y' = r/-y = -\csc\theta.$$

49. Functions of $(180°-\theta)$ and Functions of θ.—Let the student show by a geometric proof that:

$$\sin(180°-\theta) = \sin\theta, \qquad \cot(180°-\theta) = -\cot\theta,$$
$$\cos(180°-\theta) = -\cos\theta, \qquad \sec(180°-\theta) = -\sec\theta,$$
$$\tan(180°-\theta) = -\tan\theta, \qquad \csc(180°-\theta) = \csc\theta.$$

Thus: The corresponding functions of supplementary angles are equal numerically but opposite in sign, except that the respective sines and cosecants are the same both in sign and magnitude.

50. Functions of $(90°+\theta)$ and Functions of θ.—Let the student verify:

$$\sin(90°+\theta) = \cos\theta, \qquad \cot(90°+\theta) = -\tan\theta,$$
$$\cos(90°+\theta) = -\sin\theta, \qquad \sec(90°+\theta) = -\csc\theta,$$
$$\tan(90°+\theta) = -\cot\theta, \qquad \csc(90°+\theta) = \sec\theta.$$

51. Functions of $(180°+\theta)$ and Functions of θ.—Let the student verify:

$$\sin(180°+\theta) = -\sin\theta, \qquad \cot(180°+\theta) = \cot\theta,$$
$$\cos(180°+\theta) = -\cos\theta, \qquad \sec(180°+\theta) = -\sec\theta,$$
$$\tan(180°+\theta) = \tan\theta, \qquad \csc(180°+\theta) = -\csc\theta.$$

52. Functions of $(270°+\theta)$ and Functions of θ.—Verify the following relations:

$$\sin(270°+\theta) = -\cos\theta, \qquad \cot(270°+\theta) = -\tan\theta,$$
$$\cos(270°+\theta) = \sin\theta, \qquad \sec(270°+\theta) = \csc\theta,$$
$$\tan(270°+\theta) = -\cot\theta, \qquad \csc(270°+\theta) = -\sec\theta.$$

53. Generalization.—Let Fig. 27 (1) represent any angle θ generated by the radius OP, the ordinate of P being a and the abscissa b.

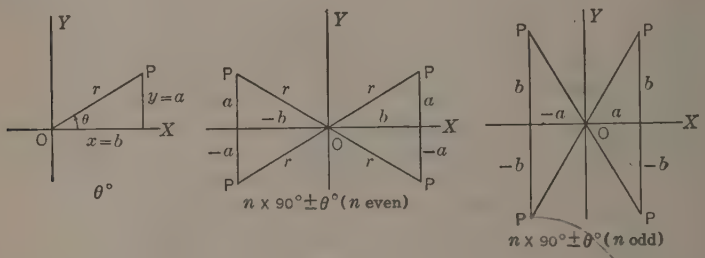


Fig. 27

If n is any even integer the radius OP will come to rest in one of the positions shown in Fig. 27 (2) after it has generated the angle $n\times90°+\theta$ or the angle $n\times90°-\theta$. For each position the magnitudes of the ordinate and abscissa are respectively the same as for the angle θ. Thus it follows that any function of $n\times90°\pm\theta$ (n even) is numerically equal to the same function of θ.

If n is any odd integer the radius OP will come to rest in one of the positions shown in Fig. 27 (3) after it has generated the angle $n\times90°+\theta$ or the angle $n\times90°-\theta$. In each position the ordinate is equal to b in magnitude and the abscissa is equal to a in magnitude. Thus it follows that any function of $n\times90°\pm\theta$ (n odd) is equal numerically to the same named cofunction of θ.

The sign before the function of θ, it appears from Fig. 27, is that of the function of $n\times 90°\pm\theta$.

Thus, the results of Arts. 47–52 may be summarized by the following rules:

1. *Any function of* $(N\times 90°\pm\theta)$, *where* N *is odd, is numerically equal to the cofunction of* θ.

2. *Any function of* $(N\times 90°\pm\theta)$, *where* N *is even, is numerically equal to the same function of* θ.

3. *The sign of the resulting function of* θ *is the sign of the original function for the quadrant of* $(N\times 90°\pm\theta)$. Since these results have been proved for all values of θ, we may *assume* $\theta<90°$ *for the purpose of determining the quadrant of* $(N\times 90°\pm\theta)$ *and the resulting sign will be correct for all values of* θ.

To illustrate: $\tan(180°+\theta)$ is numerically equal to $\tan\theta$ by (**2**). If $\theta < 90°$ then $(180°+\theta)$ is in the third quadrant where the tangent is plus; therefore, $\tan(180°+\theta) = +\tan\theta$. Again, $\csc(270°+\theta)$ is numerically equal to $\sec\theta$ by (**1**); for $\theta < 90°$, $(270°+\theta)$ is in the fourth quadrant where the cosecant is negative; therefore $\csc(270°+\theta) = -\sec\theta$.

EXERCISE XXI

1. Take θ equal to about 400°, draw a figure and derive geometrically the relations between the functions of $(270°+\theta)$ and functions of θ.

2. Drawing $\theta = 120°$ approximately, derive the relations between functions of $(180°-\theta)$ and functions of $(270°+\theta)$.

3. What is the numerical value of
$\sin 210°\tan 225°+\cos 300°\cot 315°+\tan 675°\sec 540°+\cot 495°\csc 450°$? −

4. Simplify:

$$\frac{\sin(-A)}{\cos(90°+A)}+\frac{\cos(-A)}{\sin(90°+A)}+\frac{\tan(-A)}{\cot(90°+A)}.$$

5. Simplify:

$$\frac{\tan\,(180°+x)}{\tan\,(180°-x)}\cdot\frac{\cot\,(270°-x)}{\cot\,(270°+x)}\cdot\frac{\sec\,(360°-x)}{\csc\,(360°+x)}.$$

6. Prove graphically:

(*a*) $\cot 214° = \cot 34°$, (*b*) $\sin 147° = \cos 57°$.

7. What is the value of:

$$\cot\frac{11\pi}{6}-2\cos\frac{4\pi}{3}-\frac{3}{4}\sec^2\frac{3\pi}{4}-4\sin^2\frac{5\pi}{6}?$$

8. Simplify the following:

$$\sin\left(\frac{\pi}{2}+x\right)\cos\,(\pi-x)\cot\left(\frac{3\pi}{2}+x\right)$$
$$-\sin\left(\frac{\pi}{2}-x\right)\sin\left(\frac{3\pi}{2}-x\right)\cot\left(\frac{\pi}{2}+x\right).$$

9. If $90° < A < 180°$ prove geometrically:

(*a*) $\sec\,(A-180°) = -\sec A$.
(*b*) $\tan\,(A-270°) = -\cot A$.

10. Show that $\text{hav}-A = \text{hav}\ A$.

11. Show that $\text{hav}\,(180°\pm A) = 1-\text{hav}\ A$.

12. Show that $\text{hav}\,(360°-A) = \text{hav}\ A$.

54. Significance to Tables.—By using the formulae established in this chapter *any function of any angle can always be expressed as a function of a positive angle not greater than* 45°. Thus, if we have tables giving the values of the Natural Functions and the Logarithmic Functions for angles between 0° and 45° we can find the value of the function of *any* angle by first expressing the function as a function of an angle not greater than 45°. For example:

$$\sin 1192° = \sin\,(3\times360°+112°) = \sin 112° = \sin\,(90°+22°)$$
$$= \cos 22°.$$

$$\tan(-1783^\circ) = -\tan(4\times360^\circ+343^\circ) = -\tan 343^\circ$$
$$= -\tan(360^\circ-17^\circ) = \tan 17^\circ.$$
$$\sec 2585^\circ = \sec(7\times360^\circ+65^\circ) = \sec 65^\circ = \sec(90^\circ-25^\circ)$$
$$= \csc 25^\circ.$$

The tables are arranged so that the functions of angles between 0° and 180° can be taken directly from the table. A number of degrees appears at each corner of the table. Angles between 0° and 44° appear in the upper left-hand corner on the successive pages of the table, from 45° to 89° in the lower right-hand corner, from 90° to 134° lower left-hand corner, and from 135° to 179° in the upper right-hand corner. If the number of degrees in the angle is marked at the top of the page the column is chosen by the function at the top; if the number of degrees is marked at the bottom, the column is chosen by the function at the bottom. The column of minutes used is directly above or below the number of degrees; thus for angles between 0° and 45° or 90° and 135° use the column of minutes at the left-hand side; for angles between 45° and 90° or 135° and 180° use the column of minutes at the right-hand side of the page. The matter of interpolation has been described before but care must be taken to observe from the table whether the function is increasing or decreasing (that is, whether the correction is to be added or subtracted) and to give the function the proper algebraic sign. If the function is negative (−) is written before its logarithm. Thus we write log cos 135° as cos (−) 9.84949.

For 5° at the beginning of the table of Log Sines, Tangents, and Secants the proportional parts are not given, but in the column headed "Diff 1′" are given the differences between the successive mantissas, and the proportional parts have to be computed for angles within this range. For angles near 5° where the tabular difference is large, it is well to compute corrections instead of using the proportional parts, even if they are given. For example: Find the log sin 5°58′55″. From the

table the log sin $5°58' = 9.01682$; the tabular difference is 121 and the correction is $\frac{55}{60} \times 121 = 111$. Thus log sin $5°58'55''$ $= 9.01793$.

Using the proportional parts given

$5°58'$ sin	9.01682
$55''$ Diff	121
	9.01803

Hence the use of the table of proportional parts introduces an error of 10 in the logarithm. This is accounted for by the fact that the tabular difference is not the same for the entire column and an average value is used. Using the next larger minute,

$5°59'$ sin	9.01803
$5''$ Diff	11
	9.01792

Note that this interpolation is nearly correct.

It is evident that if the proportional parts are used for this part of the table, it is better to work from the nearer minute, not necessarily from the smaller one.

EXERCISE XXII

1. Verify the following:

(*a*) $N \cos 111°07'13'' = -0.36033$.
(*b*) $\log \cos 110°11'00'' = (-)\ 9.53785 - 10$.
(*c*) $\log \sec 143°42'00'' = (-)\ 0.09370$.
(*d*) $\log \csc 157°07'00'' = 0.41021$.
(*e*) $\log \tan 98°33'09'' = (-)\ 0.82279$.
(*f*) $\log \sec 133°00'51'' = (-)\ 0.16611$.
(*g*) $\log \sec 162°00'05'' = (-)\ 0.02179$.
(*h*) $\log \csc 162°00'05'' = 0.51005$.

2. Verify the following:

(*a*) If $\log \sin \theta = 9.81237 - 10$, $\theta = 40°28'48''$ or $139°31'12''$.
(*b*) If $\log \tan \theta = (-)\ 0.37231$, $\theta = 112°59'32''$ or $292°59'32''$.
(*c*) If $\log \csc \theta = (-)\ 0.37373$, $\theta = 205°1'12''$ or $334°58'48''$.
(*d*) If $\log \sec \theta = 0.23281$, $\theta = 54°11'37''$ or $305°48'23''$.

3. Find the angles x less than 360° for which

(*a*) $\log \sin x = 9.11912 - 10$.
(*b*) $\log \cos x = (-)\ 9.42902 - 10$.
(*c*) $\log \tan x = (-)\ 0.27764$.

4. Express as a function of an acute angle not greater than 45°,

$\cot 821°$, $\tan 588°$, $\cos 765°$, $\cos -985°$, $\csc 1328°$.

5. Find from tables the numerical value of

$\cos 492°$, $\sin (-875°)$, $\cos 1188°$, $\sec 1278°$, $\cot 921°$.

6. Find the numerical value of

$$\sin \frac{15\pi}{7},\ \tan \frac{16\pi}{3},\ \csc \left(\frac{3\pi}{2} - \frac{\pi}{3}\right),\ \cos \frac{11\pi}{3}.$$

7. What is the numerical value of (to 3 decimal places) $\sin^2 x - \cos^2 x + 2 \tan x - \sec^2 x$, if $x = \dfrac{13\pi}{5}$?

8. If $a = 69°28'32''$ what is the value of

$$\frac{2 \sin 4a - \sin 10a + \sin 2a}{2 \cos 4a - \cos 10a + \cos 2a}?$$

9. If $\alpha = 201°$, $\beta = 305°$, $\gamma = 407°$ verify

$$\sin (\beta + \gamma - \alpha) + \sin (\gamma + \alpha - \beta) + \sin (\alpha + \beta - \gamma) - \sin (\alpha + \beta + \gamma) = 4 \sin \alpha \sin \beta \sin \gamma.$$

10. If $\theta = 95°57'49''$ verify the following:

$$\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 15\theta = 4 \cos 4\theta \cos 5\theta \cos 6\theta.$$

11. If $x = 29°27'13''$ use table of natural cosines to find to 4 decimals the value of

$$\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x.$$

12. If $x = 42°28'43''$ find the value of

$$\frac{\sin 3x + 2 \sin 5x + \sin 7x}{\cos 3x + 2 \cos 5x + \cos 7x}.$$

13. If $\theta = 71°29'14''$ find correct to 4 decimals the value of

$$\frac{\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta}.$$

14. Find x from

$$\tan x = \cos 8°37'55'' \times \sec 76°59'55'' \times \tan 31°47'50''.$$

15. Find the value of

$$\sin 118°26'37'' \times \cos 22°37'51'' \times \csc 94°17'25'' \times \sec 114°25'46''.$$

55. Two Important Limits.—Let θ be an acute angle and in a unit circle construct the Fig. 28.

Then $AB = \sin \theta$, $BC = \theta$, $BD = \tan \theta$.

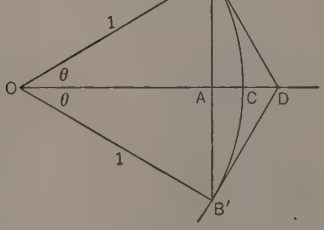


FIG. 28

From the figure

$2AB < 2BC < 2BD$

and $AB < BC < BD$,

or, $\sin \theta < \theta < \tan \theta$.

Dividing by $\sin \theta$, we obtain

$$1 < \frac{\theta}{\sin \theta} < \sec \theta;$$

and dividing by $\tan \theta$,

$$\cos \theta < \frac{\theta}{\tan \theta} < 1.$$

As θ decreases and approaches 0, $\sec \theta$ and $\cos \theta$ each approach 1. Therefore, $\lim_{\theta = 0} \frac{\theta}{\sin \theta} = 1$, and $\lim_{\theta = 0} \frac{\sin \theta}{\theta} = 1$

Similarly, $$\lim_{\theta=0} \frac{\tan\theta}{\theta} = 1.$$

The inequality shows us that the circular measure of an acute angle always lies between its sine and its tangent. For small angles the three quantities are nearly equal. The following table indicates this relation.

Angle degrees	$\sin\theta$	θ in radians	$\tan\theta$
1″	.000004848136811+	.000004848136811+	.000004848136811+
1′	.0002908882045+	.0002908882086+	.0002908882168+
1°	.0174524064+	.0174532925+	.0174550649+
5°	.08715+	.08726+	.08748+

The following formulae, obtained from the relations of this article,

$\log\sin\theta = \log\theta' + \log\sin 1'$, and $\log\tan\theta = \log\theta' + \log\tan 1'$,

where θ' denotes the number of minutes in the angle, give more accurate results than the usual interpolation within an interval of 48′ of the beginning of Table 44. These formulae can be used to find the log sine, log tangent, log cosecant, log cotangent, when the angle is less than 48′ or between 179°12′ and 180°; and also to find the log cosine, log tangent, log secant, and log cotangent when the angle is between 89°12′ and 90°48′.

The following examples illustrate their use:

1. Find log sin 22′20″.

22′20″ = 22.333′	log 1.34895
1′	sin 6.46373 − 10
0°22′20″	sin or tan 7.81268 − 10

Also log cot θ = log csc θ = log 1 − log sin θ = 2.18732.

2. Find log cot 89°49′37″.

cot 89°49′37″ = cot (90° − 0°10′23″) = tan 0°10′23″

10′23″ = 10.383′	log 1.01633
1′	tan 6.46373
89°49′37″	cot = cos 7.48006 − 10

Also, log tan θ = log sec θ = log 1 − log cos θ = 2.51994.

3. Find θ if $\log \tan \theta = 7.71259 - 10$.

$$\log \theta' = \log \tan \theta - \log \tan 1'.$$

θ	tan	$7.71259-10$
$1'$	colog	3.53627
$\theta = 17.736'$	log	1.24886

$\theta = 0°17'44''$.

EXERCISE XXIII

1. Find log sin 3°8′56″, log tan 1°26′12″.
2. Find log sec 89°42′34″ and log cot 89°42′34″.
3. Find log sine and log tangent of 21′21″.
4. Find log cosecant and log cotangent of 12′18″.
5. Find log sine and log tangent of 9′25″.
6. Find log cosecant and log cotangent of 179°41′38″.
7. Find log secant and log tangent of 90°19′22″.
8. Find θ, given $\log \tan \theta = (-)\ 7.79313 - 10$.
9. Find θ, given $\log \cos \theta = (-)\ 8.11617 - 10$.
10. Find θ, given $\log \sin \theta = 7.07831 - 10$.
11. Find θ, given $\log \csc \theta = 2.72836$.
12. Find θ, given $\log \sec \theta = (-)\ 2.11134$.

56. The Sine.—We shall now investigate the behavior of the sine, as the angle takes on different values. In making this investigation we first plot a number of points, for each of which the abscissa is a value of the angle and the ordinate its sine, and then draw a smooth curve through the points. This curve is known as the sine curve or graph of the sine.

It is possible to obtain points on the sine curve by a simple construction without estimating the numerical values of their coördinates. By definition $\sin \theta = \frac{y}{r}$; if $r = 1$, then $\sin \theta = \frac{y}{1} = y$;

that is, the ordinate y in a unit circle represents the value of the sine.

At equal intervals on the circumference of a unit circle (in Fig. 29 the interval used is 15° or $\pi/12$ radians) draw ordinates. Then lay off $2\pi = 6.28$ units on the axis of abscissas and subdivide into 24 equal parts; each part representing $360°/24 = 15°$ $= 0.26$ radians approx. At each point so found erect an ordinate equal to the corresponding ordinate of the unit circle as shown in the figure.

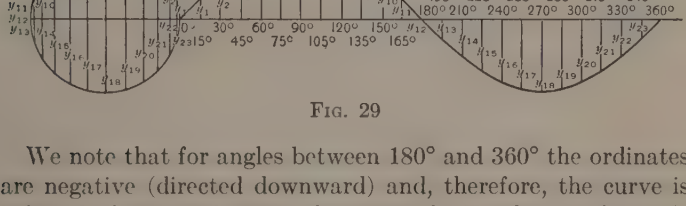


Fig. 29

We note that for angles between 180° and 360° the ordinates are negative (directed downward) and, therefore, the curve is below the horizontal axis in this interval. Further, as the angle increases continuously from 0° to 90°, the sine increases continuously from 0 to 1; as the angle increases from 90° to 180° the sine decreases from 1 to 0; as the angle increases from 180° to 270° the sine decreases from 0 to -1; as the angle increases from 270° to 360° the sine increases from -1 to 0.

The radius vector has completed a revolution during this increase in the angle from 0° to 360°. For a second revolution —an increase in the angle from 360° to 720°—or any revolution in which the angle increases from $n\times360°$ to $(n+1)\times360°$, the sine will take the same series of values, starting at 0, increasing to 1, decreasing to 0, decreasing to -1 and increasing to 0. Graphically, this means that the curve repeats itself every 2π (360°) units or that any two ordinates 2π or $n\times2\pi$ apart are equal. Because of this property the sine is called a *periodic*

function. The interval by which the angle must increase before the cycle of values of the function begins to repeat is called the period. Thus the period of the sine is 2π radians or 360°.

57. The Tangent.—To find a single line representing the value of tan θ we shall substitute a new ratio for y/x. Thus in a unit circle (Fig. 30) by similar triangles $\tan\theta = y/x = z \quad r = z$, where z is the segment of a tangent line drawn to the circle at A, which is cut off by the radius extended. If $\theta = n \times 360° + a$ where $90° < a < 270°$ the radius must be extended backward as shown in positions (2) and (3) of the figure. If we add the condition that z be considered positive when extending upward and negative when extending downward we shall also obtain the correct sign.

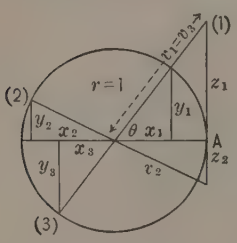


FIG. 30

To construct the graph of the tangent, divide the circumference of the unit circle at equal intervals, as in the previous cases, draw the line representing the value of the tangent of each angle, and plot. (See Fig. 31.)

The graph shows that the tangent starts at 0° and increases continuously as the angle increases continuously from 0° to 90°. When the angle reaches 90° the value of the tangent has passed beyond any finite magnitude. We say the tangent of 90° is infinity and write $\tan 90° = +\infty$. If we let the angle decrease from 180° to 90°, the tangent starts at 0 and decreases, and when the angle reaches 90° the tangent has become infinite, always being negative. In this case we write $\tan 90° = -\infty$. Thus, if the angle is generated by positive rotation we write $\tan 90° = +\infty$, and if the angle is generated by negative rotation $\tan 90° = -\infty$, but in general we write $\tan 90° = \infty$.

We may describe the variation in the value of the tangent as follows: as the angle increases from 0° to 90° the tangent increases from 0 to ∞; as the angle increases from 90° to 180°

the tangent increases from $-\infty$ to 0; as the angle increases from 180° to 270°, the tangent increases from 0 to $+\infty$; as the

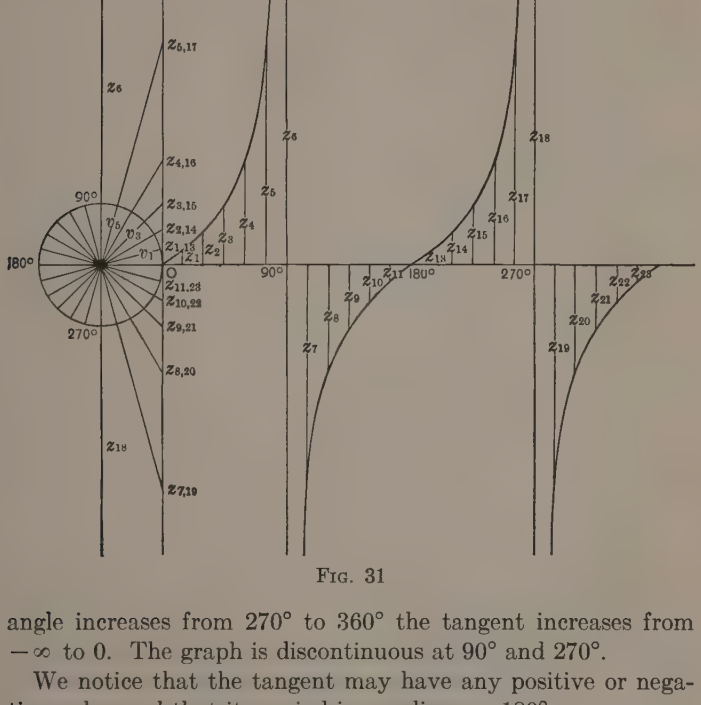


FIG. 31

angle increases from 270° to 360° the tangent increases from $-\infty$ to 0. The graph is discontinuous at 90° and 270°.

We notice that the tangent may have any positive or negative value and that its period is π radians or 180°.

EXERCISE XXIV

1. Using the table of Art. 25 to find the values of the ordinate, construct the graph of $\cos x$, taking values of x which differ by 15°. As x increases from 0° to 360°, discuss the variation in $\cos x$.

2. Construct the graph of cot x, using the table of Art. 25 to find the values of the ordinate. Discuss the variation in cot x as x increases from 0° to 360°.

3. Construct the graph of sec x and discuss the variation in sec x as x increases from 0° to 360°.

4. Construct the graph of csc x and discuss the variation in csc x as x increases from 0° to 360°.

5. Show that if a sheet of paper, rolled into a circular cylinder, is cut by a plane oblique to the elements of the cylinder and then unrolled, the cut edge forms a sine curve.

58. Summary.—The range of values covered by the trigonometric functions is as follows:

$$-1 \leqq \sin x \text{ or } \cos x \leqq +1,$$
$$-1 \leqq \sec x \text{ or } \csc x \leqq +1,$$
$$-\infty \leqq \tan x \text{ or } \cot x \leqq +\infty.$$

As x increases the variation of the six functions is shown by the following table:

Quadrant	I	II	III	IV
sin x	0 to +1	+1 to 0	0 to −1	−1 to 0
cos x	+1 to 0	0 to −1	−1 to 0	0 to +1
tan x	0 to +∞	−∞ to 0	0 to +∞	−∞ to 0
cot x	+∞ to 0	0 to −∞	+∞ to 0	0 to −∞
sec x	+1 to +∞	−∞ to −1	−1 to −∞	+∞ to +1
csc x	+∞ to +1	+1 to +∞	−∞ to −1	−1 to −8

The *period* of the sine, cosine, secant, or cosecant is 360° or 2π radians. The period of the tangent or cotangent is 180° or π radians.

CHAPTER VII

FUNCTIONS OF TWO ANGLES; MULTIPLE ANGLES; SUBMULTIPLE ANGLES

59. The Sine, Cosine, and Tangent of the Sum of Two Angles.—Let α and β be two positive acute angles such that $\alpha+\beta<90°$.

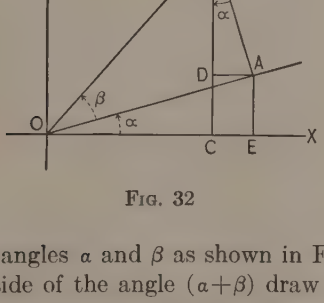


FIG. 32

Construct the angles α and β as shown in Fig. 32. From B in the terminal side of the angle $(\alpha+\beta)$ draw a perpendicular, BA to OA, the terminal side of α; also draw a perpendicular, BC to OX; from A draw the perpendiculars AD and AE to BC and OX respectively. $\angle CBA = \angle \alpha$, since its sides are respectively perpendicular to the sides of α.

Then $$\sin(\alpha+\beta)=\frac{CB}{OB}=\frac{CD+DB}{OB}=\frac{EA+DB}{OB}=\frac{EA}{OB}+\frac{DB}{OB},$$

multiplying the first fraction, numerator and denominator, by OA and the second by AB

$$\sin(\alpha+\beta)=\frac{EA}{OA}\times\frac{OA}{OB}+\frac{BD}{BA}\times\frac{AB}{OB},$$

or $$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta. \tag{9}$$

Also, $$\cos(\alpha+\beta)=\frac{OC}{OB}=\frac{OE-CE}{OB}=\frac{OE}{OB}-\frac{DA}{OB},$$

$$=\frac{OE}{OA}\times\frac{OA}{OB}-\frac{DA}{AB}\times\frac{AB}{OB},$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta. \tag{10}$$

Dividing (9) by (10),

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta},$$

dividing each term of numerator and denominator by $\cos\alpha\cos\beta$,

$$=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}},$$

or $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}. \tag{11}$$

In the derivation of the formulae (9) and (10) it was assumed that α and β are positive angles such that $\alpha+\beta<90°$, but it is not difficult to show that these formulae hold for positive or negative angles of any size.

60. Other Allied Formulae.—From (9) and (10) we obtain:

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta, \tag{12}$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta, \tag{13}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}. \tag{14}$$

By making $\beta=\alpha$ in formulae (9), (10), and (11) we obtain:

$$\sin 2\alpha=2\sin\alpha\cos\alpha, \tag{15}$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha, \tag{16}$$

or $$\cos 2\alpha=1-2\sin^2\alpha, \tag{17}$$

or
$$\cos 2\alpha = 2\cos^2\alpha - 1, \tag{18}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}. \tag{19}$$

By making $\beta = 2\alpha$ in formulae (15) to (19), we find:

$$\sin^2\frac{\beta}{2} = \frac{1-\cos\beta}{2} = \text{hav}\,\beta, \tag{20}$$

$$\cos^2\frac{\beta}{2} = \frac{1+\cos\beta}{2}, \tag{21}$$

$$\tan\frac{\beta}{2} = \pm\sqrt{\frac{1-\cos\beta}{1+\cos\beta}} = \frac{1-\cos\beta}{\sin\beta} = \frac{\sin\beta}{1+\cos\beta}. \tag{22}$$

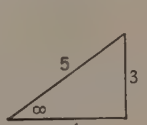
The student should verify formulae (12) to (22)..

61. *Examples.*—**1.** Find the sine of 75°.

$$\sin 75° = \sin(45°+30°) = \sin 45°\cos 30° + \cos 45°\sin 30°,$$
$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

2. If $\sin\beta = \frac{1}{3}$ and $\cos\alpha = \frac{4}{5}$ where α and β are both acute find $\cos(\alpha+\beta)$ and $\tan(\alpha-\beta)$.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$= \frac{4}{5}\times\frac{\sqrt{8}}{3} - \frac{3}{5}\times\frac{1}{3} = \frac{8\sqrt{2}-3}{15}.$$



$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{3}{4}-1/\sqrt{8}}{1+\frac{3}{4}\times 1/\sqrt{8}}$$

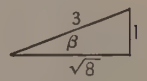


FIG. 33

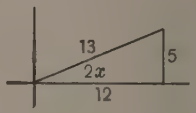
$$= \frac{6\sqrt{2}-4}{8\sqrt{2}+3} = \frac{108-50\sqrt{2}}{119}.$$

3. Show that $\sin 7x \cos 4x - \cos 7x \sin 4x = \sin 3x$.

Note that left-hand number is the expansion of $\sin(7x-4x) = \sin 3x$.

4. If $\sin 2x = \frac{5}{13}$, find $\sin x$, $\cos x$, $\tan x$. If $\sin 2x = \frac{5}{13}$ we have from the figures:

(a) If $2x$ is in the first quadrant, $\cos 2x = \frac{12}{13}$.



Then, $\sin x = \sqrt{\dfrac{1-\frac{12}{13}}{2}} = 1/\sqrt{26}$,

$\cos x = \sqrt{\dfrac{1+\frac{12}{13}}{2}} = 5/\sqrt{26}$,

$\tan x = \dfrac{1/\sqrt{26}}{5/\sqrt{26}} = \dfrac{1}{5}$.

(b) If $2x$ is in the second quadrant, $\cos 2x = -\frac{12}{13}$.

Then, $\sin x = \sqrt{\dfrac{1+\frac{12}{13}}{2}} = 5/\sqrt{26}$,

$\cos x = \sqrt{\dfrac{1-\frac{12}{13}}{2}} = 1/\sqrt{26}$,

$\tan x = \dfrac{5/\sqrt{26}}{1/\sqrt{26}} = 5$.

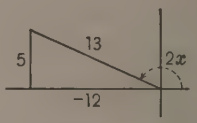


FIG. 34

5. Show that

$$\frac{\sin 2a + \sin a}{1 + \cos a + \cos 2a} = \tan a.$$

If $$\frac{\sin 2a + \sin a}{1 + \cos a + \cos 2a} = \tan a,$$

then, $$\frac{(2 \sin a \cos a) + \sin a}{(2 \cos^2 a) + \cos a} = \tan a,$$

$$\frac{\sin a\,(2 \cos a + 1)}{\cos a\,(2 \cos a + 1)} = \tan a,$$

$$\tan a = \tan a.$$

Therefore, $$\frac{\sin 2\alpha+\sin \alpha}{1+\cos \alpha+\cos 2\alpha}=\tan \alpha.$$

EXERCISE XXV

1. Make a table of values of the functions of 15°, 75°, 105° by using the formulae of this chapter. Check by using relations of Art. 53.

2. If $\tan \alpha=\frac{4}{3}$ and $\cot \beta=\frac{5}{12}$, find $\cot (\alpha+\beta)$, $\sin (\alpha-\beta)$, $\cos (\alpha-\beta)$. ($\alpha<90°$, $\beta<90°$.)

3. If $\tan \alpha=\frac{5}{6}$ and $\cot \beta=11$, show that $\alpha+\beta=45°$, when α and β are acute.

4. If $\tan \alpha=\frac{1}{2}$ and $\tan \beta=\frac{1}{3}$, find the functions of $(2\alpha+\beta)$. ($\alpha<90°$, $\beta<90°$.)

5. Derive a formula for the $\sin (\alpha+\beta+\gamma)$ in terms of the sines and cosines of these angles.

6. Show that $\sin (60°+y)-\sin y=\sin (60°-y)$.

7. Show that $\sin 105°=\cos 45°-\cos 105°$.

8. Prove

$$\sin (45°-x) \cos (45°-y)-\cos (45°-x) \sin (45°-y) = -\sin (x-y).$$

9. Show that

$$\cos (30°+\alpha) \cos (30°-\alpha)+\sin (30°+\alpha) \sin (30°-\alpha)=\cos 2\alpha.$$

10. Prove $\cos (\alpha+\beta) \cos (\alpha-\beta)=\cos^2 \alpha-\sin^2 \beta$.

11. Prove $\sin (x+y) \sin (x-y)=\cos^2 y-\cos^2 x$.

12. Derive the formulae for $\tan (\alpha+\beta)$ from Fig. 32.

13. Show that $\tan (45°+\theta)=\dfrac{1+\tan \theta}{1-\tan \theta}$.

14. Show that $\tan (45°-\theta)=\dfrac{1-\tan \theta}{1+\tan \theta}$.

15. Prove $\tan(45°+x)\tan(135°+x)=-1$.

16. Prove

$$\sin(60°+x)\cos(30°+x)-\cos(60°+x)\sin(30°+x)=1/2.$$

17. Prove $\dfrac{\tan 2a+\tan a}{1-\tan 2a\tan a}=\tan 3a$.

18. Prove $\dfrac{\tan 5a-\tan 2a}{1+\tan 5a\tan 2a}=\dfrac{\tan 2a+\tan a}{1-\tan a\tan 2a}$.

19. Prove $\csc 2\theta=\cot\theta-\cot 2\theta$.

20. Prove $\dfrac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)}=\dfrac{\cot\alpha+\cot\beta}{1+\cot\alpha\cot\beta}$.

21. Prove $\dfrac{\sin(B-C)}{\cos B\cos C}+\dfrac{\sin(C-A)}{\cos C\cos A}+\dfrac{\sin(A-B)}{\cos A\cos B}=0$.

22. Prove

$$\cos 4x\cos x-\sin 4x\sin x=\cos 3x\cos 2x-\sin 3x\sin 2x.$$

23. If $A+B+C=90°$, show that

$$\sin^2 A+\sin^2 B+\sin^2 C+2\sin A\sin B\sin C=1.$$

24. Derive formulae for the $\cot(\alpha+\beta)$ and $\cot(\alpha-\beta)$ from figures.

25. Do the same analytically by the method used in Art. 59 for $\tan(\alpha+\beta)$.

26. Derive the formulae (12), (13), and (14) from a figure.

27. Derive formulae for $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$ respectively.

28. Derive formulae for $\sin 4\theta$ and $\cos 4\theta$ in terms of functions of θ.

29. Derive by means of the half-angle formulae the values of the functions of $22\frac{1}{2}°$ from the functions of $45°$.

30. Derive the values of functions 15° from the functions of 30°.

31. If $\sin\theta = 1/3$, find $\cos 2\theta$, $\sin 3\theta$, $\tan 4\theta$. $(\theta < 90°.)$

32. If $\tan\theta = 1/2$, find $\tan 2\theta$, $\tan 3\theta$, $\cos 3\theta$. $(\theta < 90°.)$

33. Prove $\cos^4\theta - \sin^4\theta = \cos 2\theta$.

34. Prove $\cot\theta - \tan\theta = \dfrac{2}{\tan 2\theta}$.

35. Prove $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta$.

36. Prove $\dfrac{\cos\theta}{1-\sin\theta} = \tan\left(45° + \dfrac{\theta}{2}\right)$.

37. Prove $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2$.

38. Prove $\dfrac{\cos(15° + a)}{\sin(15° + a)} + \dfrac{\sin(15° - a)}{\cos(15° - a)} = \dfrac{4\cos 2a}{1 + 2\sin 2a}$

39. Prove $\cos^2(45° - a) - \sin^2(45° - a) = \sin 2a$.

40. Prove $\dfrac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \tan\theta$.

41. Prove $4(\cos^3 20° + \cos^3 40°) = 3(\cos 20° + \cos 40°)$.

42. Prove $\dfrac{\tan a}{\tan a - \tan 3a} + \dfrac{\cot a}{\cot a - \cot 3a} = 1$.

43. Prove $\dfrac{1}{\sin 10°} - \dfrac{\sqrt{3}}{\cos 10°} = 4$.

44. Prove $2 \operatorname{hav} A = \dfrac{\tan^2 A}{\sec A + 1 + \tan^2 A}$.

If $\sin A = \dfrac{4}{5}$, $\tan B = \dfrac{5}{12}$ and $\cos A$ and $\cos B$ are negative, find

45. $\sin B/2$, $\tan 2A$, $\tan(A + B)$.

46. $\cos(B-A)$, $\cot(2A+B)$.

47. $\tan 4A$, $\sin 3A$, $\tan A/2$.

62. Sums or Differences of Sines or Cosines as Products.— From formulae (9) and (12) we have

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$$

Adding these equations we get

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta. \tag{23}$$

Subtracting them

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta. \tag{24}$$

Adding the formulae (10) for $\cos(\alpha+\beta)$ and (13) for $\cos(\alpha-\beta)$ we obtain

$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta. \tag{25}$$

Subtracting the second from the first

$$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta. \tag{26}$$

These formulae are equally useful when transformed.

Let $\alpha+\beta=\theta$ and $\alpha-\beta=\theta'$, then solving simultaneously $\alpha=\dfrac{\theta+\theta'}{2}$, $\beta=\dfrac{\theta-\theta'}{2}$; and we may write the above formulae as

$$\sin\theta+\sin\theta'=2\sin\frac{\theta+\theta'}{2}\cos\frac{\theta-\theta'}{2}. \tag{27}$$

$$\sin\theta-\sin\theta'=2\cos\frac{\theta+\theta'}{2}\sin\frac{\theta-\theta'}{2}. \tag{28}$$

$$\cos\theta+\cos\theta'=2\cos\frac{\theta+\theta'}{2}\cos\frac{\theta-\theta'}{2}. \tag{29}$$

$$\cos\theta-\cos\theta'=-2\sin\frac{\theta+\theta'}{2}\sin\frac{\theta-\theta'}{2}. \tag{30}$$

The formulae (23) to (30) transform a sum or difference of two sines or two cosines into products, or by reversing them

express a product of two sines, two cosines, or a sine and a cosine as sums. By successive applications we can effect the transformation of sums of more than two terms and products of more than two factors, as will be illustrated in the next article.

63. *Examples.*—**1.** Express $\sin 3x+\sin 13x$ as a product.

By formula (27)

$$\sin 3x+\sin 13x=2\sin\frac{3x+13x}{2}\cos\frac{3x-13x}{2},$$

$$=2\sin 8x\cos 5x.$$

2. Express $\cos x\cos 2x\cos 3x$ as a sum.

By reversing (25) we may write

$$\cos x\cos 2x\cos 3x=\cos 2x\,[\tfrac{1}{2}\cos(3x+x)+\tfrac{1}{2}\cos(3x-x)],$$

$$=\tfrac{1}{2}\cos 2x\cos 4x+\tfrac{1}{2}\cos^2 2x,$$

$$=\tfrac{1}{2}\,[\tfrac{1}{2}\cos(4x+2x)+\tfrac{1}{2}\cos(4x-2x)]+\tfrac{1}{2}\cos^2 2x, \text{ by (25)}$$

$$=\tfrac{1}{4}\cos 6x+\tfrac{1}{4}\cos 2x+\tfrac{1}{4}(1+\cos 4x), \text{ by (21)}$$

$$=\tfrac{1}{4}(\cos 6x+\cos 4x+\cos 2x+1).$$

3. Simplify:

$$\frac{\cos x+2\cos 5x+\cos 9x}{\sin x+2\sin 5x+\sin 9x},$$

$$\frac{2\cos 5x+\cos 9x+\cos x}{2\sin 5x+\sin 9x+\sin x}=\frac{2\cos 5x+2\cos\dfrac{9x+x}{2}\cos\dfrac{9x-x}{2}}{2\sin 5x+2\sin\dfrac{9x+x}{2}\cos\dfrac{9x-x}{2}}$$

$$=\frac{2\cos 5x+2\cos 5x\cos 4x}{2\sin 5x+2\sin 5x\cos 4x}=\frac{2\cos 5x\,(1+\cos 4x)}{2\sin 5x\,(1+\cos 4x)}=\cot 5x.$$

EXERCISE XXVI

Simplify:

1. $\sin 11x+\sin 7x$, $\cos 4x-\cos 2x$, $\sin 33\pi/7-\sin 19\pi/7$.

2. Prove $\dfrac{\sin 70^\circ+\sin 50^\circ}{\sin 80^\circ}=\sqrt{3}$.

3. Prove $\dfrac{\sin 2x-\sin 2y}{\sin 2x+\sin 2y}=\tan (x-y)\cot (x+y)$.

4. Prove $\sin (60°+x)-\sin (60°-x)$

$$=\frac{1}{\sqrt{3}}\tan x\,[\cos (30°-x)+\cos (30°+x)].$$

5. Prove $\sin 70°-\sin 50°-\sin 10°=0$

6. Prove $\dfrac{\cos 5x-\cos 3x}{\sin 5x-\sin 3x}+\dfrac{\cos 2x-\cos 4x}{\sin 4x-\sin 2x}=\dfrac{-\sin x}{\cos 4x\cos 3x}$.

7. Prove $\cos 3x/2+\cos 5x/2+\cos 7x/2+\cos 15x/2$

$$=4\cos 2x\cos 5x/2\cos 3x.$$

8. Prove $\cos 140°+\cos 100°+\cos 20°=0$.

9. Prove

$\sin 10°+\sin 20°+\sin 40°+\sin 50°-\sin 70°-\sin 80°=0$.

10. Prove $\dfrac{\sin (\alpha+\beta)-2\sin \alpha+\sin (\alpha-\beta)}{\cos (\alpha+\beta)-2\cos \alpha+\cos (\alpha-\beta)}=\tan \alpha$.

11. Simplify

$$\frac{\cos(\alpha+\beta+\gamma)+\cos(-\alpha+\beta+\gamma)+\cos(\alpha-\beta+\gamma)+\cos(\alpha+\beta-\gamma)}{\sin(\alpha+\beta+\gamma)+\sin(-\alpha+\beta+\gamma)-\sin(\alpha-\beta+\gamma)+\sin(\alpha+\beta-\gamma)}.$$

12. Prove $8\sin 20°\sin 40°\sin 80°=\sqrt{3}$.

13. Prove $\cos 20°\cos 40°\cos 60°\cos 80°=\frac{1}{16}$.

14. Prove $\dfrac{\sin (N-1)\alpha+\sin (N+1)\alpha+2\sin N\alpha}{\cos (N-1)\alpha-\cos (N+1)\alpha}=\cot \dfrac{\alpha}{2}$.

15. Prove $\cos^2 p\theta-\cos^2 (p-1)\theta=-\sin (2p-1)\theta\sin \theta$.

16. Prove $\cos 3x\sin 2x-\cos 4x\sin x=\cos 2x\sin x$.

17. Prove $\sin x\cos (x+y)-\cos x\sin (x-y)=\cos 2x\sin y$.

18. Prove $4\sin x\sin (60°+x)\sin (60°-x)=\sin 3x$.

19. Prove $2\cos 3\pi/11\cos 4\pi/11+\cos 4\pi/11+\cos 10\pi/11=0$.

20. Prove $\dfrac{\cos 3x \sin 2x - \cos 4x \sin x}{\cos 5x \cos 2x - \cos 4x \cos 3x} = -\cot 2x.$

If $\alpha+\beta+\gamma=180°$, prove

21. $\sin 2\alpha+\sin 2\beta+\sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma.$

22. $\cos \alpha+\cos \beta-\cos \gamma = -1+4 \cos \alpha/2 \cos \beta/2 \sin \gamma/2.$

23. $\sin^2 \alpha+\sin^2 \beta-\sin^2 \gamma = 2 \sin \alpha \sin \beta \cos \gamma.$

24. $\dfrac{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}{\sin \alpha+\sin \beta+\sin \gamma} = 8 \sin \alpha/2 \sin \beta/2 \sin \gamma/2.$

25. $\tan \dfrac{\alpha}{2} \tan \dfrac{\beta}{2}+\tan \dfrac{\beta}{2} \tan \dfrac{\gamma}{2}+\tan \dfrac{\gamma}{2} \tan \dfrac{\alpha}{2}=1.$

26. $\cos \dfrac{\alpha}{2}+\cos \dfrac{\beta}{2}+\cos \dfrac{\gamma}{2}$

$$= 4 \cos\left(\frac{\alpha+\beta}{4}\right) \cos\left(\frac{\beta+\gamma}{4}\right) \cos\left(\frac{\gamma+\alpha}{4}\right).$$

27. $\cot \dfrac{\alpha}{2}+\cot \dfrac{\beta}{2}+\cot \dfrac{\gamma}{2} = \cot \dfrac{\alpha}{2} \cot \dfrac{\beta}{2} \cot \dfrac{\gamma}{2}.$

28. $\dfrac{\sin \alpha+\sin \beta+\sin \gamma-\sin (\alpha+\beta+\gamma)}{\cos \alpha+\cos \beta+\cos \gamma+\cos (\alpha+\beta+\gamma)}$

$$= \tan \frac{\alpha+\beta}{2} \tan \frac{\beta+\gamma}{2} \tan \frac{\gamma+\alpha}{2}.$$

29. State the formulae (23) to (26) in words.

30. State the formulae (27) to (30) in words.

CHAPTER VIII

THE OBLIQUE TRIANGLE

64. The Sine Law.—Let A, B, and C denote the angles of any plane triangle and a, b, c the respective opposite sides. Circumscribe a circle about the triangle and denote the radius by R and its center by O. Draw the radii OB, OC and drop the perpendicular, OD upon a, as shown in Fig. 35. Fig. 35 (a) shows a triangle all of whose angles are acute; Fig. 35 (b) shows a triangle with an obtuse angle.

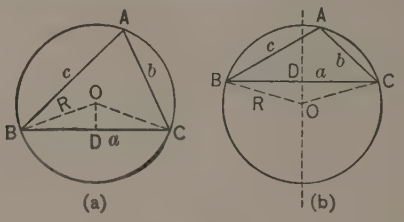


FIG. 35

From the construction $\angle BOD = 1/2 \angle BOC$.

Fig. 35 (a), $\angle A = 1/2 \angle BOC$ since $\angle BOC$ is measured by arc BC, and $\angle A$ by 1/2 the same arc;

Fig. 35 (b), $\angle A = 1/2(360° - \angle BOC)$.

Therefore, $\angle A = \angle BOD$, Fig. 35 (a);

$\angle A = 180° - \angle BOD$, Fig. 35 (b);

and $$\sin A = \sin \angle BOD = \frac{BD}{R}.$$

Then, $$BD = R \sin A,$$

and $$a = 2BD = 2R \cdot \sin A.$$

Similarly, by dropping perpendiculars from O upon b and c, we can show

$$b = 2R \cdot \sin B,$$
$$c = 2R \cdot \sin C.$$

Solving the last three equations for $2R$ and equating the values, we have

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad (31)$$

The relation indicated by these equations is known as the Sine Law, and is stated in words as follows: *The sides of a plane triangle are proportional to the sines of the opposite angles.* The common ratio is equal to the diameter of the circumscribed circle.

65. Applications of the Sine Law to the Oblique Triangle.—Case 1. *Given: One Side and Two Angles.* The third angle is found with the aid of the relation $A+B+C=180°$, and the two unknown sides by applications of the Sine Law.

Example.—In a plane triangle, given: $c=227.22$, $C=70°16'49''$, $A=47°23'18''$; find the remaining parts.

$$B=180°-(A+C)=62°19'53''$$

From the Sine Law	$a=c\cdot\csc C\cdot\sin A$,	$b=c\cdot\csc C\cdot\sin B$.
$c=227.22$	log 2.35645	log 2.35645
$C=70°16'49''$	csc 0.02625	csc 0.02625
$A=47°23'18''$	sin 9.86685	
$B=62°19'53''$		sin 9.94726
$a=177.64$	log 2.24955	
$b=213.78$		log 2.32996

66. Case II.—*Given: Two Sides and an Angle Opposite One of Them.* From geometry we know that when two sides and an angle opposite one of them are given, the triangle is not always uniquely determined. These parts may have such values that the construction cannot be made, or such that one triangle can be constructed, or again such that two can be constructed.

Letting the given parts be a, c and A, then, when A is obtuse there will be one solution provided $a>c$, otherwise no solution. When A is acute, there is one solution if $a\geq c$ or if $a=c\cdot\sin A$

(the length of the perpendicular dropped from the end of c upon the unknown side); there is no solution if $a<c \sin A$; there are two solutions if $a<c$ and at the same time $a>c \sin A$ (See Fig. 36). To determine the relation between a

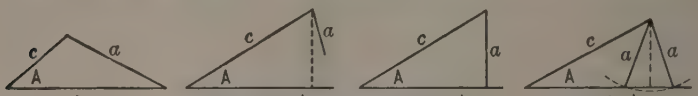


FIG. 36

and $c \cdot \sin A$, it is convenient to note that $\sin C = \frac{c}{a} \sin A$ by the Sine Law, and, therefore, $\log \sin C$ would be positive if $a < c \sin A$; that $\log \sin C = 0$ when $a = c \cdot \sin A$; and that $\log \sin C$ is negative when $a > c \cdot \sin A$.

Example.—In a plane triangle $b = 237.33$, $a = 213.72$, $A = 60°$; find the remaining parts.

Since A is acute and $a < b$, there will be two solutions if $\log \sin B$ is negative. The computation can be arranged as follows:

	$\sin B = \frac{b}{a} \sin A$	$c_1 = a \sin C_1 \csc A$	$c_2 = a \sin C_2 \csc A$
$b = 237.33$	log 2.37535		
$a = 213.72$	colog 7.67016	log 2.32984	log 2.32984
$A = 60°00'00''$	sin 9.93753	csc 0.06247	csc 0.06247
$B_1 = 74°05'30''$	sin 9.98304		
$B_2 = 105°54'30''$			
$C_1 = 45°54'30''$		sin 9.85626	
$C_2 = 14°05'30''$			sin 9.38645
$c_1 = 177.24$		log 2.24857	
$c_2 = 60.084$			log 1.77876

Since B is determined from its sine, two values less than 180° can be found and thus the double solution is introduced analytically.

Example: In a plane triangle $a = 49.366$, $b = 26.437$, $A = 47° 18'36''$; find the remaining parts.

Since $b < a$ then $B < A$ and there is only one solution. The computation follows:

	$\sin B = \frac{b}{a} \sin A$	$c = a \sin C \csc A$
$a = 49.366$	colog 8.30657	log 1.69343
$b = 26.437$	log 1.42221	
$A = 47°18'36''$	sin 9.86631	csc 0.13369
$B = 23°10'51''$	sin 9.59509	
$C = 109°30'33''$		sin 9.97433
$c = 63.307$		log 1.80145

EXERCISE XXVII

Find the remaining parts of each of the following triangles:

1. $B = 69°30'12''$, $A = 66°39'24''$, $a = 598.29$.
2. $A = 118°41'29''$, $B = 25°08'37''$, $c = 7.2348$.
3. $a = 378.43$, $C = 110°28'26''$, $B = 38°40'50''$.
4. $A = 117°24'33''$, $B = 32°11'27''$, $c = 31.435$.
5. $C = 68°29'15''$, $B = 45°24'18''$, $c = 4449$.
6. Determine the number of solutions in each of the following cases:

   (*a*) $A = 30°$, $c = 58$, $a = 29$.
   (*b*) $B = 30°$, $c = 42$, $b = 14$.
   (*c*) $B = 30°$, $c = 45$, $b = 30$.
   (*d*) $C = 69°30'$, $a = 12.76$, $c = 15.00$.
   (*e*) $A = 60°$, $c = 910$, $a = 776$.
   (*f*) $B = 123°40'$, $c = 219$, $b = 207$.

7. $c = 392.0$, $a = 273.0$, $A = 37°14'$.
8. $b = 635.75$, $c = 285.75$, $C = 20°14'25''$.
9. $b = 22.143$, $a = 12.932$, $A = 47°24'32''$.
10. $c = 78.493$, $B = 66°12'10''$, $b = 71.818$.
11. $a = 54.287$, $b = 31.082$, $A = 124°49'12''$.
12. $b = 318.35$, $a = 275.19$, $A = 47°19'30''$.
13. $c = 139.06$, $b = 115.97$, $B = 43°11'32''$.
14. $b = 75.013$, $a = 74.201$, $A = 37°23'43''$.
15. $a = 0.09476$, $b = 0.19873$, $B = 127°12'10''$.
16. Derive the Sine Formula by dropping a perpendicular

from any vertex to the opposite side and expressing its value in terms of the sines of the other two angles.

67. The Tangent Formula.—From Art. 64, we know

$$a/b = \sin A/\sin B$$

and by composition and division we obtain

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

But $$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

Therefore,

$$\frac{\mathbf{a-b}}{\mathbf{a+b}} = \frac{\mathbf{\tan\frac{1}{2}(A-B)}}{\mathbf{\tan\frac{1}{2}(A+B)}}. \tag{32}$$

In like manner it can be shown that

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)} \text{ and } \frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

68. From the Sine Formula also, we obtain

$$(a+b)/c = (\sin A + \sin B)/\sin C,$$

$$= \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}C\cos\frac{1}{2}C},$$

and since $\frac{C}{2} = 90° - (A+B)/2$ we may write,

$$= \frac{\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A+B)} = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)},$$

or,

$$\frac{\mathbf{a+b}}{\mathbf{c}} = \frac{\mathbf{\cos\frac{1}{2}(A-B)}}{\mathbf{\cos\frac{1}{2}(A+B)}}. \tag{33}$$

Similarly,

$$\frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C},$$

$$= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}C\cos\frac{1}{2}C};$$

and

$$\frac{\mathbf{a-b}}{\mathbf{c}} = \frac{\mathbf{\sin\frac{1}{2}(A-B)}}{\mathbf{\sin\frac{1}{2}(A+B)}}. \tag{34}$$

69. Case III.—*Given: Two Sides and the Included Angle.* The formulae (32) to (34) enable us to solve this case. Let a, b and C be the given parts; then $(A+B)/2 = 90° - C/2$. By the Tangent Law we can find $(A-B)/2$, which added to $(A+B)/2$ gives A, and subtracted from $(A+B)/2$ gives B; c can be determined by either (33) or (34) but a check is obtained by using both.

Example.—In a plane triangle $a=4.469$, $b=3.357$, $C=69°\ 28'17''$; find the remaining parts.

A form for the computation and the results are shown below.

$$\tan\frac{A-B}{2} = \tan\frac{A+B}{2}\cdot\frac{a-b}{a+b}\ (1),\quad c=(a+b)\cos\frac{A+B}{2}\sec\frac{A-B}{2}\ (2),$$

$$c=(a-b)\sin\frac{A+B}{2}\csc\frac{A-B}{2}\ (3).$$

		(1)	(2)	(3)
$a=4.469$				
$b=3.357$				
$a+b=7.826$	colog		log	
$a-b=1.112$	log			log
$\frac{A+B}{2}=55°15'52''$	tan	_____	cos	sin
$\frac{A-B}{2}=11°34'52''$	tan 9.31160		sec _____	csc _____
$A=66°50'44''$				
$B=43°41'00''$				
$c=4.552$			log 0.65819	log 0.65819

EXERCISE XXVIII

Find the remaining parts of the triangle, given:

1. $a=232.23$, $b=169.82$, $C=81°13'24''$.
2. $a=292.27$, $c=195.59$, $B=65°18'32''$.
3. $b=232.23$, $c=195.59$, $A=61°13'00''$.
4. $a=234.7$, $b=185.4$, $C=84°36'00''$.
5. $a=122.71$, $c=194.23$, $B=116°42'12''$.

6. $b=2579.3$, $c=5287.4$, $A=138°21'12'$.
7. $a=3.8947$, $b=5.2918$, $C=67°12'20''$.
8. $a=0.59312$, $b=0.22734$, $C=64°38'00''$.

70. The Cosine Law.—In any oblique triangle, as shown in Fig. 37, let p be the length of the perpendicular let fall from C upon c, or c produced.

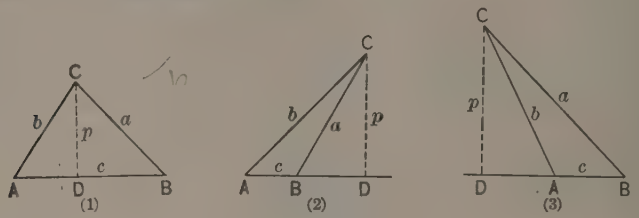


FIG. 37

(1) $AD=b\cos A$. $BD=c-DA=c-b\cdot\cos A$.
(2) $AD=b\cdot\cos A$. $BD=AD-c=b\cdot\cos A-c$.
(3) $AD=b\cdot\cos(180°-A)=-b\cdot\cos A$ $BD=c+AD=c-b\cdot\cos A$.

For each triangle we have

$$b^2-b^2\cos^2 A=p^2=a^2-(c-b\cos A)^2.$$

Simplifying and solving for a we have

$$\mathbf{a^2=b^2+c^2-2bc\cdot\cos A.} \qquad (35)$$

In a similar manner, dropping perpendiculars from A and B, we can show

$$b^2=a^2+c^2-2ac\cdot\cos B$$

and

$$c^2=a^2+b^2-2ab\cdot\cos C.$$

The relation indicated by these formulae is called the Cosine Law, and may be stated as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of the two sides and the cosine of the included angle.

Example.—Find the angles of the triangle in which $a=9$, $b=11$, $c=13$.

Using the Cosine Law:

$$\cos A = \frac{11^2+13^2-9^2}{2\times 11\times 13} = \frac{209}{286} = 0.731, \quad A = 43.0°,$$

$$\cos B = \frac{9^2+13^2-11^2}{2\times 9\times 13} = \frac{129}{234} = 0.551, \quad B = 56.6°,$$

$$\cos C = \frac{9^2+11^2-13^2}{2\times 9\times 11} = \frac{33}{198} = 0.167, \quad C = 80.4°.$$

$$A+B+C = 180°.$$

EXERCISE XXIX

(Use Table of Art. 25.)

1. Find the angles of the triangle whose sides are 6, 7, and 8.

2. Find the angles of the triangle whose sides are 13, 14, and 16.

3. Find the third side of the triangle of which two sides are 13 and 11 and the included angle is 37°.

4. Find the third side of the triangle in which $a=5$, $b=12$, and $C=117°$.

5. Find the perimeter of the triangle in which $b=40$, $c=60$, and $A=63°$.

6. Find the angles of the triangle in which $a=60$, $b=40$, $c=35$.

7. Find the largest angle of the triangle in which $a=65$, $b=58$, $c=83$.

8. Two adjacent sides of a parallelogram are 27 feet and 11 feet and the included angle is 41°30′. Find the length of each diagonal of the parallelogram.

71. The formulae of Art. 70 are not convenient for logarithmic computation; hence, from them we derive others to which logarithms can be easily applied. Substituting $1-2\sin^2 A/2$ for $\cos A$ (by (17)) in formula (35) we obtain,

$$a^2 = b^2 + c^2 - 2bc\,(1 - 2\sin^2 A/2),$$
$$= b^2 + c^2 - 2bc + 4bc \sin^2 A/2.$$

Solving for $\sin^2 A/2$, we have

$$\sin^2 \frac{A}{2} = \frac{a^2 - (b^2 + c^2 - 2bc)}{4bc} = \frac{a^2 - (b-c)^2}{4bc} = \frac{(a-b+c)(a+b-c)}{4bc}.$$

If we put $a+b+c=2s$, then $a-b+c=2(s-b)$ and $a+b-c=2(s-c)$; also by (21), $\sin^2 A/2 = \text{hav } A$. Substituting equal values we obtain

$$\textbf{hav A} = \sin^2 \textbf{A}/2 = (\textbf{s}-\textbf{b})\ (\textbf{s}-\textbf{c})/\textbf{bc}. \qquad (36)$$

Similarly, $\text{hav } B = \sin^2 B/2 = (s-a)\ (s-c)/ac,$
$\text{hav } C = \sin^2 C/2 = (s-a)\ (s-b)/ab.$

Again, $\cos^2 A/2 = 1 - \sin^2 A/2 = 1 - (s-b)\ (s-c)/bc,$

$$= \frac{bc - s^2 + bs + cs - bc}{bc} = \frac{s(b+c-s)}{bc},$$

and since $b+c-s = 2s-a-s = s-a$, we have

$$\cos^2 \textbf{A}/2 = \textbf{s}(\textbf{s}-\textbf{a})/\textbf{bc}. \qquad (37)$$

Similarly, $\cos^2 B/2 = s(s-b)/ac,$
$\cos^2 C/2 = s(s-c)/ab.$

72. We can now derive

$$\tan^2 \textbf{A}/2 = \frac{\sin^2 A/2}{\cos^2 A/2} = (\textbf{s}-\textbf{b})(\textbf{s}-\textbf{c})/\textbf{s}(\textbf{s}-\textbf{a}). \qquad (38)$$

Similarly, $\tan^2 \dfrac{B}{2} = \dfrac{(s-a)(s-c)}{s(s-b)},$

$$\tan^2 \frac{C}{2} = \frac{(s-a)(s-b)}{s(s-c)}.$$

In any triangle let r be the radius of the inscribed circle, x the distance from the vertex C to an adjacent point of tangency, and y and z the corresponding distances from A and B respectively, as shown in Fig. 38. Then

$$2s=a+b+c=2x+2y+2z \quad \text{or} \quad s=x+y+z,$$

$$a=x+z,$$
$$b=x+y,$$
$$c=y+z.$$

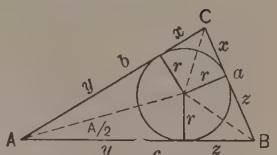


FIG. 38

Solving for x, y and z, we find

$$y=s-(x+z)=s-a, \quad x=s-c, \quad z=s-b.$$

From the figure $\tan\dfrac{A}{2}=\dfrac{r}{y}=\dfrac{r}{(s-a)}$, $\tan\dfrac{B}{2}=\dfrac{r}{(s-b)}$,

$$\tan\frac{C}{2}=\frac{r}{(s-c)}. \tag{39}$$

But from formula (38)

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Therefore, $$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \tag{40}$$

which is the radius of the inscribed circle.

73. Case IV.—*Given the Three Sides.* When the three sides of a triangle are given, any of the formulae (36) to (39) may be used to find the angles. The formula for the tangent (39), considering all cases, is more accurate than the sine (36) and cosine (37) formulae and requires taking only four logarithms from the table to find all the angles. The haversine formula (36), also, gives a very convenient solution, as will be shown in the next article, and in most cases gives greater accuracy than any of the other formulae.

Example.—Find the angles of the triangle, given: $a=365.74$, $b=445.84$, $c=545.62$.

The solution is as follows:

$$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\ (1),\quad \tan\frac{A}{2}=\frac{r}{(s-a)}\ (2),$$

$$\tan\frac{B}{2}=\frac{r}{(s-b)}\ (3),\quad \tan\frac{C}{2}=\frac{r}{(s-c)}\ (4).$$

		(1)	(2)	(3)	(4)
$a=$	365.74				
$b=$	445.84				
$c=$	545.62				
$2s=$	1357.20				
$s=$	678.60	colog 7.16839			
$s-a=$	312.86	log 2.49535	colog 7.50465		
$s-b=$	232.76	log 2.36691		colog 7.63309	
$s-c=$	132.98	log 2.12378			colog. 7.87622
$s=$	678.60 2	log 4.15443			
r	.....	log 2.07722	log 2.07722	log 2.07722	log 2.07722
$A=41°47'46''$,		$\frac{A}{2}=20°53'53''$	tan 9.58187		
$B=54°20'12''$,		$\frac{B}{2}=27°10'06''$		tan 9.71031	
$C=$	$83°52'08''$ / $180°00'06''$	$\frac{C}{2}=41°56'04''$			tan 9.95344

The sum $A+B+C$ should be nearly equal to 180°, but rarely comes out exactly so because of the error in the final figure of the logarithm. (See Art. 43.) Even if the sum does come out to be exactly 180° it does not necessarily follow that the individual angles are exact.

EXERCISE XXX

Find the angles in each of the following triangles:

1.	$a=61.043$,	$b=49.207$,	$c=80.516$.
2.	$a=35$,	$b=38$,	$c=41$.
3.	$a=7.9302$,	$b=5.0771$,	$c=4.8309$.
4.	$a=282.78$,	$b=312.36$,	$c=196.94$.
5.	$a=97.862$,	$b=105.98$,	$c=138.72$.
6.	$a=57.872$,	$b=50.103$,	$c=34.982$.
7.	$a=0.5862$,	$b=0.98713$,	$c=1.0352$.
8.	$a=95.321$,	$b=113.72$,	$c=179.84$.

74. The Haversine Table.—Haversine tables are being more and more widely used, especially by the navigator and the engineer. The table of "Haversines" gives both the logarithmic and natural (*in heavy type*) haversines. Tables of proportional parts are not given and the interpolations are made in the usual way, as is shown by the following examples.

In Table 45 of Bowditch Useful Tables, the sexagesimal minutes are printed in heavy type and not on every line, and the plus sign in front indicates that these minutes are to be added to the number of minutes at the top of the column of haversines. For the first 15° the function is tabulated for every 30″, from 15° to 120° for every 15″, from 120° to 135° for every 30″; and, of course, in interpolation the Tab. Diff. of 15″ or 30″, as the case may be, should be used instead of the Tab. Diff. for 1′.

1. Find the log haversine and N haversine of 5°37′23″.

From the table	5°37′30″	N hav 0.00241	log hav 7.38159
	5°37′00″	N hav 0.00240	log hav 7 38030
Tab. Diff.	30″	1	129

Corrections to be added $\frac{23}{30}\times 1 = 0.4$, $\frac{23}{30}\times 129 = 99$.

Therefore, we find 5°37′23″ N hav 0.00240 log hav 7.38129.

2. Find x if log hav $x = 8.77118$.

From the table,	log hav 28°7′45″	$= 8.77127$
	log hav 28°7′30″	$= 8.77114$
Tab. Diff.	15″	13

Correction to be added $\frac{4}{13}\times 15'' = 5''$. Therefore, $x = 28°7'35''$.

Since the log haversine which occurs on the same line with the N haversine is the logarithm of that N haversine, the

"Haversine" table can be used as a table of logarithms of numbers. However, it is well not to use the first part, but begin where the N haversine is approximately 0.10000.

The solution by the haversine formulae (36) of the example in Art. 73 is given below.

$$\text{hav } A = \frac{(s-b)(s-c)}{bc} (1), \quad \text{hav } B = \frac{(s-a)(s-c)}{ac} (2), \quad \text{hav } C = \frac{(s-a)(s-b)}{ab} (3).$$

	(1)	(2)	(3)
$a =$ 365.74		colog 7.43683	colog 7.43683
$b =$ 445.84	colog 7.35082		colog 7.35082
$c =$ 545.62	colog 7.26311	colog 7.26311	
$2s =$ 1357.20			
$s =$ 678.60			
$s-a =$ 312.86		log 2.49535	log 2.49535
$s-b =$ 232.76	log 2.36691		log 2.36691
$s-c =$ 132.98	log 2.12378	log 2.12378	
$s =$ 678.60			
$A = 41°47'45''$	hav 9.10462		
$B = 54°20'09''$		hav 9.31907	
$C = 83°52'05''$			hav 9.64991
$179°59'59''$			

EXERCISE XXXI

1. Verify the following tables, first using the angle to find the value of the function, secondly finding the angle from the function:

Angle	*Log hav*	*Angle*	*Nat. hav*
5°59′52″	7.43728.	6°59′00″	0.00371.
71°55′18″	9.53762.	8°36′30″	0.00563.
119°44′22″	9.87392.	93°17′41″	0.52874.
23°57′40″	8.63437.	135°40′18″	0.85767.

2. Find the log haversine of 18′45″, 28′53″, 37′48″, 48′25″.

3. Solve the triangles of Exercise XXX by the haversine formulae (36).

4. Given $\text{hav}X = \text{hav}(a-b) + \cos a \cdot \cos b \cdot \text{hav}c$, find X when $a = 47°27'46''$, $b = 17°49'13''$, $c = 37°27'54''$.

5. Given $\text{hav}Y = \sin(s-a) \cdot \sin(s-b) \cdot \sec a \cdot \sec b$ and $s = (a+b+c)/2$, find Y when $a = 23°47'25''$, $b = 31°24'58''$, $c = 99°47'34''$.

6. Show that in a right triangle $\text{hav}A = (c-b)/2c$. This is a useful formula when the given parts b and c are very nearly equal.

7. Solve the right triangle, given $c = 483.9832$, $b = 481.6751$ by the formula of Ex. 6.

8. Solve the right triangle, given $b = 678.9974$, $c = 679.0843$ by the formula of Ex. 6.

75. Denoting the area of the triangle by K, we have from Fig. 37

$$K = \tfrac{1}{2}\, pc = \tfrac{1}{2}\, b \cdot c \cdot \sin A, \tag{41}$$

which is a formula for the area when two sides and the included angle are given.

From the Sine formula we have $c = b \sin C / \sin B$; substituting in (41), we obtain

$$K = \frac{b^2 \sin A \sin C}{2 \sin B} = \frac{b^2 \sin A \sin C}{2 \sin (A+C)}, \tag{42}$$

which gives the area in terms of a side and two adjacent angles.

From Fig. 38 we have

$$K = \tfrac{1}{2}\, r \cdot (a+b+c) = r \cdot s = \sqrt{s(s-a)(s-b)(s-c)}, \tag{43}$$

formulae which give the area in terms of the three sides and the radius of the inscribed circle.

From Art. 64, $\sin A = a/2R$, and substituting in (41) we have

$$K = \frac{abc}{4R}, \tag{44}$$

which gives the area in terms of the three sides and the radius of the circumscribed circle.

EXERCISE XXXII

Find K, r, and R for the following six triangles:

1. $a=23.762$, $b=19.823$, $c=25.921$.

2. $a=282.78$, $b=312.36$, $c=196.94$.

3. $a=25.783$, $b=37.686$, $c=50.404$.

4. $a=785.62$, $b=679.35$, $c=324.87$.

5. $a=16$, $b=17$, $c=19$.

6. $a=6.8235$, $b=5.2063$, $c=3.1628$.

7. A triangle whose sides are proportional to 3, 4, and 5 is inscribed in a circle of radius 10 in. Find the sides and area of the triangle.

8. A triangle whose sides are proportional to 4, 5, and 6 is inscribed in a circle of radius 12 in. Find the sides and the area.

9. Find the area of the triangle in which $a=24.932$, $b=31.754$, $C=81°22'00''$.

10. Find the area of the triangle, given $b=229.37$, $c=352.73$, $A=60°12'00''$.

76. In the preceding articles the formulae necessary for the solution of the four possible cases arising in the solution of the oblique triangle have been derived. For each case the given parts and formulae are as follows:

1. One side and two angles—The Sine formula. (31)
2. Two sides and the angle opposite one of them—The Sine formula. (31)
3. Two sides and the included angle—The tangent formula and special formula for the third side. (32), (33), (34).
4. Three sides—Formulae for the tangents of the half-angles or formulae for the haversines of the angles. (36), (39).

In the problems in the miscellaneous set the student should draw the figure for the problem and write the formulae necessary for the solution. He should then make a complete form for the logarithmic computation and finally use the tables.

EXERCISE XXXIII

Miscellaneous Problems

1. In an oblique triangle given $a=68.721$, $b=37.683$, $c=54.247$; find A, B, C, r, R, and K.

2. Solve the triangle, given $b=49.001$, $a=31.329$, $A=32°\ 18'37''$.

3. Solve the triangle, given $c=231.97$, $a=136.19$, $C=137°\ 48'18''$.

4. Given $A=18°54'28''$, $C=109°48'27''$, $c=489.45$; find a, b and B.

5. Given $a=0.36842$, $b=0.79436$, $c=0.89284$; find A, B, and C.

6. Given $C=69°28'14''$, $a=29.431$, $b=43.249$; find A, B, and c.

7. Given $a=641.29$, $A=31°29'14''$, $b=1219.4$; find B, C and c.

8. Given $A=139°28'14''$, $b=641.27$, $a=1794.7$; find B, C and c.

9. Given $a=64.94$, $b=74.45$, $c=84.23$; find A, B and C.

10. Two cruisers start from the same point at 18 and 10 knots respectively, the first travelling due East and the second due Northeast. Find the distance between them at the end of two hours and the bearing of each from the other.

11. Two sides of a parallelogram are 102.73 and 74.32, and one of the diagonals is 154.21. Find the other diagonal and the angles of the parallelogram.

12. Three posts are the vertices of a triangle ABC on the level ground with $AB=250$ ft., $BC=312$ ft., and $CA=345$ ft. If the posts are of heights 32, 40 and 32 ft., respectively, find the area of the triangle of which their tops are the vertices.

13. From two points, one on either side of a tower, and 453 ft. apart, the angles of elevation of its top are $48°12'$ and $52°18'$ respectively. What is the height of the tower?

14. From a ship two lighthouses bear due Northeast. After sailing two hours at 11 knots on a course of 130° (Navy Compass) the lighthouses bear 6° and 356° respectively. What is the distance between the lighthouses?

15. It is desired to measure the width of a river. A tree is observed on the opposite side at A. From B, a point directly opposite A, a distance of 400 ft. is measured off perpendicular to AB to the point C. The angle BCA is found to be 36°25′12″. What is the width of the river?

16. The angles of elevation of a blimp, measured at two points on level ground and in the same vertical plane as the blimp, are 47°12′ and 20°46′. The two points are a half mile apart and on the same side of the blimp; what is its height?

17. To measure an inaccessible distance AB as a base line CD is laid off. If CD is 500 ft., and the angles are measured to be $CDB = 78°12'$, $CDA = 46°18'$, $DCA = 99°36'$, and $DCB = 61°43'$, all in the same plane, what is the length AB?

18. Two objects are separated by an obstruction. A third point accessible to both objects is chosen and the distance from it to each of the objects is measured. These distances are 984 ft. and 623 ft. and the included angle is 94°13′. What is the distance between the objects?

19. The angle of depression of an object viewed from the top of a tower is 35°22′51″ and the angle of depression of a second object 427 ft. nearer and in the same straight line with the first object and the foot of the tower is 67°47′18″. What is the height of the tower?

20. From a point at the foot of a hill surmounted by a tower, the angle of elevation of the top of the tower is 48°32′. After going 1000 ft. up the hill the tower subtends an angle of 72°. The hill makes an angle of 15°20′ with the horizontal. How far is the top of the tower above the horizontal plane of the foot of the hill?

21. To measure the height of an inaccessible tower, AB, a line CD, which is 540.8 ft. long is measured off in the horizontal plane through, A, the foot of the tower and the following angles measured: $ACB = 55°48'$, $BCD = 58°43'$, $CDB = 50°32'$. Find the height of the tower.

22. Two points A and B are on opposite sides of a river. A straight line CD is run through A and the following measurements taken: $CA = 649$ ft., $AD = 427$ ft., $\angle BCA = 48°29'$, $\angle BDA = 54°21'$. Find AB.

23. To find the inaccessible distance AB, a third point C is chosen and the following measurements taken: $AC = 500$ ft., $\angle BAC = 74°28'$, $\angle BCA = 45°26'$. Find AB.

24. From two points P and Q 1450 ft. apart in a horizontal plane, observations of a balloon A, are taken at the same instant. At P the angle of elevation of A is $73°29'$, at Q it is $48°27'$; the angle $APQ = 40°29'$ and $\angle AQP = 30°27'$. Find the height of the balloon.

25. Show that the area of a segment of a circle of radius R is $\frac{1}{2}R^2(\theta - \sin\theta)$ where θ is the angle of the segment. In a circle of radius 15 ft. find the area of the smaller segment cut off by a chord of 11 ft.

26. A belt is stretched around two pulleys of radius 2 ft. and 3 ft., respectively, the centers of which are 12 ft. apart. Find the length of the belt when crossed; when not crossed.

27. Find the area and perimeter of a regular dodecagon inscribed in a circle of radius 18.3 ft.

28. The length of a lake subtends at a certain point $47°18'$, and the distances from this point to the two ends of the lake are 5890 ft. and 6728 ft. Find the length of the lake.

29. Two sides of a parallelogram are 187.4 ft. and 149.8 ft., and the included angle is $49°28'$. Find the area.

30. A tower makes an angle of $113°28'$ with the inclined plane on which it stands; and at a distance of 124 ft. from its

base, measured down the plane, the angle subtended by the tower is 23°27′. Find the height of the tower.

31. The sides of a triangle are 189 ft., 210 ft., and 78 ft., find the length of the longest median.

32. From a ship two rocks are observed to be in the same line with the ship, bearing north. After the ship has sailed N 60° W for 6 miles, the first rock bears N 75° E, and the second N 30° E. Find the distance between the rocks.

33. From a window on the level with the bottom of a tower the angle of elevation of the top is 48°29′, and from a second window 25 ft. higher the angle of elevation is 39°13′. Find the height of the tower.

34. At two stations, the height of a kite subtends the same angle, namely 47°20′. The angle which the line joining one station and the kite subtends at the other station is 58°22′, the distance between the stations 1829 feet. Find the height of the kite.

35. The diagonals of a quadrilateral are 814.4 and 567.3 ft. and intersect at an angle of 37°24′. Find the area of the quadrilateral.

CHAPTER IX

TRIGONOMETRIC EQUATIONS AND INVERSE FUNCTIONS

77. General Expressions for All Angles Having the Same Value for a Given Function.—From the definition of the trigonometric ratios it is evident that a given angle can have but one value for its sine, one for its cosine, etc. However, the converse of this is not true, as we have already found. In general for each of the trigonometric ratios we have learned that for any given value of the function there are two angles less than 360°. In discussing the graphs of the functions, this idea was somewhat extended. We shall now seek a general expression for the angles for which a function has a given value.

78. General Expression for All Angles that Have the Same Value for the Sine: for the Cosecant.—Let us illustrate this case by means of the equation $\sin\theta=\frac{1}{2}$. If positive and less than 360°, θ can be either 30° or 150° as noted before. These two values of θ give two distinct positions for the radius vector; i.e., one position 30° above the x-axis on the right, another 30° above the x-axis on the left. As the revolving radius vector continues to sweep around in a counter-clockwise direction it will pass through both of these positions during each complete revolution. Let us tabulate the values of θ.

1st revolution	(1) $0\cdot\pi+\frac{\pi}{6}$,	(2) $1\cdot\pi-\frac{\pi}{6}$,
2d "	(3) $2\cdot\pi+\frac{\pi}{6}$,	(4) $3\cdot\pi-\frac{\pi}{6}$,
3d "	(5) $4\cdot\pi+\frac{\pi}{6}$,	(6) $5\cdot\pi-\frac{\pi}{6}$,
4th "	(7) $6\cdot\pi+\frac{\pi}{6}$,	(8) $7\cdot\pi-\frac{\pi}{6}$,

5th " (9) $8\cdot\pi+\frac{\pi}{6}$, (10) $9\cdot\pi-\frac{\pi}{6}$,

etc.

At each of these positions the sin θ will be equal to $\frac{1}{2}$. It will be noticed that to even multiples of π the $\frac{\pi}{6}$ is added and that from odd multiples the $\frac{\pi}{6}$ is subtracted. An expression which is positive for n even and negative for n odd is $(-1)^n$. Hence we may write

$$\theta = n\,\pi+(-1)^n\frac{\pi}{6}.$$

We have discussed this for n positive or zero; the same value of θ will be obtained for n negative. The discussion is left to the student.

We shall generalize the above result and say "*The expression* $n\pi+(-1)^n\alpha$ *is a general expression for all angles which have the same sine as the angle* α."

Since the cosecant is the reciprocal of the sine this expression applies also to it.

79. Cosine and Secant.—We have learned that $2\pi-\alpha$ is the only angle between 0 and 2π which has the same value for the cosine as α. Here our sequence, as the revolving line moves counter-clockwise, will be

$$\alpha,\ 2\pi-\alpha,\ 2\pi+\alpha,\ 4\pi-\alpha,\ 4\pi+\alpha, \text{ etc.}$$

As it moves clockwise

$$-\alpha,\ -2\pi+\alpha,\ -2\pi-\alpha,\ -4\pi+\alpha,\ -4\pi-\alpha, \text{ etc.}$$

or $$\theta = 2n\pi\pm\alpha.$$

The expression $2n\pi\pm\alpha$ *is a general expression for all angles which have the same value for the cosine as the angle* α.

Since the secant is the reciprocal of the cosine, this expression applies also to it.

80. Tangent and Cotangent.—The angle $\pi+\alpha$ is the only angle between 0 and 2π which has the same value for the tangent as α.

Here our sequence will be

$$\alpha,\ \pi+\alpha,\ 2\pi+\alpha,\ 3\pi+\alpha,\ 4\pi+\alpha, \text{ etc.};$$
$$-\pi+\alpha,\ -2\pi+\alpha,\ -3\pi+\alpha,\ -4\pi+\alpha, \text{ etc.}$$

or $$\theta=n\pi+\alpha.$$

The expression $n\pi+\alpha$ is a general expression for all angles which have the same value for the tangent as the angle α.

Since the cotangent is the reciprocal of the tangent the expression applies also to it.

81. *Examples.*—In the illustrative examples following the *smallest positive value of* α will be used. It is not necessary to use this as other expressions equally true can readily be given but we shall do so in order to obtain uniform results.

1. Given $\sin\theta=-1/\sqrt{2}$; what is the general expression for θ?

The smallest positive value of θ is $225°$ or $\frac{5\pi}{4}$.

Hence $$\theta=n\pi+(-1)^n\frac{5\pi}{4}.$$

This result could be written $\theta=n\pi-(-1)^n\frac{\pi}{4}$.

It is left to the student to demonstrate that these two results are the same.

2. Given $\cos\theta=0$. What is the general expression for θ?

The smallest positive value of θ is $90°$ or $\frac{\pi}{2}$.

Hence $$\theta=2n\pi\pm\frac{\pi}{2}.$$

3. Given $\cot\theta=\sqrt{3}$, what is the general expression for θ?

The smallest positive value of θ is $\frac{\pi}{6}$.

Hence $$\theta=n\pi+\frac{\pi}{6}.$$

4. Given $\sin\theta=1$. What is the general expression for θ?

The smallest positive value of θ is $\frac{\pi}{2}$.

Further this is the only value of θ between 0 and 2π.

Hence in this case

$$\theta = n\pi + (-1)^n \frac{\pi}{2},$$

or

$$\theta = 2n\pi + \frac{\pi}{2}.$$

5. What is the general expression for all angles whose tangent is $1/\sqrt{3}$ and whose cosine is $-\sqrt{3}/2$?

The only angles between 0 and 2π whose tangent $= 1/\sqrt{3}$ are $\pi/6$ and $7\pi/6$; the only angles between 0 and 2π whose cosine $= -\sqrt{3}/2$ are $7\pi/6$ and $5\pi/6$. The only angle which satisfies both conditions is $7\pi/6$.

Hence the general value is

$$\theta = 2n\pi + \frac{7\pi}{6}.$$

Show that this is the same result as

$$\theta = (2n+1)\pi + \pi/6.$$

6. Find the general value of θ which satisfies the equation

$$2 \sin^2 \theta = 1.$$

Dividing by 2 and taking the square root of both sides we have

$$\sin \theta = \pm 1/\sqrt{2}.$$

From the fact that $\sin \theta = \frac{1}{\sqrt{2}}$ we obtain $\theta = n\pi + (-1)^n \frac{\pi}{4}$.

From the fact that $\sin \theta = \frac{-1}{\sqrt{2}}$ we obtain $\theta = n\pi + (-1)^n \frac{5\pi}{4}$.

These two together would constitute the general answer, according to the rules given. However, a simpler form is

$$\theta = n\pi \pm \pi/4.$$

Show that this last result is equivalent to the other two taken together.

EXERCISE XXXIV

Find the general value of the angle which satisfies each of the following equations:

1. $\sin\theta=\frac{1}{2}$.
2. $\sin x=1$.
3. $\cos y=-1/\sqrt{2}$.
4. $\sec z=-\sqrt{2}$.
5. $\cot\theta=\sqrt{3}$.
6. $\csc y=-2$.
7. $\cos a=1$.
8. $\tan x=-1$.
9. $\sin\theta=-1$.
10. $\tan x=\infty$.
11. $\sin^2 y=1$.
12. $3\tan^2 z=1$.
13. $\sec^2\theta=2$.
14. $4\sin^2 a=1$.
15. $\sec^2\phi=4$.
16. $\csc^2 x=4$.
17. $\cos^2\theta=0$.
18. $\cot^2\phi=1$.
19. $3\sec^2 x=4$.
20. $2\cos^2 z=1$.

What is the general value of x that satisfies both of the following equations?

21. $\sin x=-1, \quad \cos x=0$.
22. $\cos x=\frac{\sqrt{3}}{2}, \quad \tan x=-\frac{1}{3}\sqrt{3}$
23. $\sec x=\sqrt{2}, \quad \cot x=-1$.

82. Equations involving one or more trigonometric functions of an unknown angle are called trigonometric equations. The examples given at the end of the last section are all trigonometric equations of a simple type.

The solution of a trigonometric equation involves the finding of all values of the unknown angle which will satisfy the equation. The laws of algebra for solving equations apply and in addition the equation can be transformed by employing trigonometric identities.

Fixed rules for solving cannot be given except for particular types. Usually it is best to change all the trigonometric functions present into a single trigonometric function; factor when possible, being very careful not to lose sight of any factors and hence of any solutions. Verify the result obtained by substitution in order to eliminate any extraneous solutions.

ILLUSTRATIONS.

1. $\cos^2 x - \sin^2 x = \frac{1}{2}$.

$\cos^2 x - (1 - \cos^2 x) = \frac{1}{2}$,

$2\cos^2 x = 1 + \frac{1}{2} = \frac{3}{2}$,

$\cos^2 x = \frac{3}{4}$,

$\cos x = \pm\frac{\sqrt{3}}{2}$,

$x = n\pi \pm \frac{\pi}{6}$;

or $\cos^2 x - \sin^2 x = \frac{1}{2}$.

$\cos^2 x - \sin^2 x = \cos 2x = \frac{1}{2}$,

$2x = 2n\pi \pm \frac{\pi}{3}$,

$x = n\pi \pm \frac{\pi}{6}$.

2. $\sin\frac{x}{2} + \cos\frac{x}{2} = \sqrt{2}$.

$\sin\frac{x}{2} = \sqrt{2} - \cos\frac{x}{2}$,

$\sin^2\frac{x}{2} = 2 - 2\sqrt{2}\cos\frac{x}{2} + \cos^2\frac{x}{2}$.

$2\cos^2\frac{x}{2} - 2\sqrt{2}\cos\frac{x}{2} + 1 = 0$.

$\cos\frac{x}{2} = \frac{2\sqrt{2} \pm \sqrt{8-8}}{4} = \frac{\sqrt{2}}{2}$.

Apparently our answer here is

$\frac{x}{2} = 2n\pi \pm \frac{\pi}{4}$; $x = 4n\pi \pm \frac{\pi}{2}$.

However, if we test the

$\frac{x}{2} = 2n\pi - \frac{\pi}{4}$, we shall obtain

$-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$,

which is obviously incorrect. Hence, the proper result is

$\frac{x}{2} = 2n\pi + \frac{\pi}{4}$; $x = 4\,n\pi + \frac{\pi}{2}$.

3. $\csc^2\theta + \cot\theta = 3$.

$1 + \cot^2\theta + \cot\theta = 3$.

$\cot^2\theta + \cot\theta - 2 = 0$.

$\cot\theta = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2$ or 1.

$\theta = n\cdot 180° + 153°26'$ or $n\pi + \frac{\pi}{4}$.

4. $\sin 4x = 2 \sin 2x.$

$\sin 4x = 2 \sin 2x \cos 2x = 2 \sin 2x.$

$\sin 2x = 0,$ or $\cos 2x = 1.$

$2x = n\pi.$ $2x = 2n\pi.$

$x = \frac{n\pi}{2}.$ $x = n\pi.$

Here are apparently two results; however, all the results in $n\pi$ are included in the result $n\pi/2$; therefore, the complete result is,

$$x = \frac{n\pi}{2}.$$

The particular solutions between 0 and 2π are

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

5. $\cos 7\theta - \cos \theta + \sin 4\theta = 0.$

$-2 \sin 4\theta \cdot \sin 3\theta + \sin 4\theta = 0,$

$\sin 4\theta (2 \sin 3\theta - 1) = 0.$

$\sin 4\theta = 0.$ $\sin 3\theta = \frac{1}{2}.$

$4\theta = n\pi,$ $3\theta = n\pi + (-1)^n \frac{\pi}{6},$

$\theta = \frac{n\pi}{4}.$ $\theta = \frac{n\pi}{3} + (-1)^n \frac{\pi}{18}.$

EXERCISE XXXV

Solve the following equations: in each case give the general value of the angle and the particular values between 0 and 2π.

1. $\sin 2x - \cos x = 0.$

2. $\sin x + \csc x = 2.$

3. $2 \sin^2\theta - \cos \theta = 1.$

4. $\tan^2 y + \cot^2 y = 2.$

5. $\tan \theta \cdot \sec \theta = -\sqrt{2}.$

6. $\tan 2\theta = \tan \theta.$

7. $\cos 6\theta = \cos 2\theta.$

8. $6 \sin \theta + \cos \theta = 2.$

9. $\cos 2x - 4 \sin x = \frac{5}{2}.$

10. $\cos^2\theta - \sin \theta = \frac{1}{4}.$

11. $\cos 4\theta = \sin 2\theta.$

12. $\tan 2\theta \cdot \tan \theta = 1.$

13. $\sin 2\theta = \sin \theta.$

14. $\sin 4x - \sin 2x - \cos 3x = 0.$

15. $4 \cos 2x + 3 \cos x = 1.$

16. $\sin 3\theta = \sin 9\theta.$

17. $\sin 5x - \sin 3x + \sin x = 0.$

18. $\tan 4\theta = \cot 3\theta.$

19. $\cos \theta + \cos 2\theta + \cos 3\theta = 0.$

Hint:—$\cot 3\theta = \tan (\pi/2 - 3\theta).$

20. $\sin(x+60°) - \sin(x-60°) = -\frac{1}{2}\sqrt{3}$.

21. $\sin 7x - \sin 4x + \sin x = 0$.

22. $\sin 3\theta + \sin 7\theta = \sin 5\theta$.

Are the following identities or equations? If identities verify them; if equations, solve them.

23. $\cos^2\theta = 1 - \sin\theta$.

24. $\sec^2 x + \csc^2 x = \frac{16}{3}$.

25. $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$.

26. $\tan 2x + \sec 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x}$.

27. $\text{vers}\, y = \text{vers}\, 2y$.

28. $\cos x \csc x \tan x = \sin x \sec x \cot x$.

29. $\cot 2\theta + \tan\theta + 1 = 0$.

30. $(\tan y - \sin y)^2 + (1-\cos y)^2 = (\sec y - 1)^2$.

31. $2 \sin 3z \cos z = 2 \sin z \cos 3z - 1$.

32. $\sec 2a - \tan 2a \tan a = 1$.

Inverse Trigonometric Functions

83. Inverse Functions.—From the equation $\sin\theta = x$ we know that θ is an angle whose sine is x. In order to express $\sin\theta = x$ inversely it is customary to write $\theta = \text{arc sin}\, x$ or $\theta = \sin^{-1} x$, which is read "θ is the angle whose sine is x," or "θ is the inverse sine of x," or "θ is the anti-sine x." Thus, $\theta = \sin^{-1} x$ represents in concise form all the angles, for which the value of the sine is x. Similarly $\tan^{-1} b = \phi$ and $\tan\phi = b$, $\text{vers}\,\theta = y$ and $\theta = \text{vers}^{-1} y$ are pairs of equivalent equations.

It should be carefully observed that the -1 in this notation is not an algebraic exponent and that $\sin^{-1} a$ does not denote $\dfrac{1}{\sin a} = (\sin a)^{-1}$. (Art. 37.)

The expressions, $\sin^{-1}\frac{1}{2}$, $\text{arc cos}\, x$, $\tan^{-1} y$, $\text{arc tan}\, y$, $\text{vers}^{-1} b$ etc. are called "Inverse Trigonometric Functions."

84. Principal Values.—The *smallest positive angle* denoted by the inverse function is often called *the principal value* of the inverse function.

If ϕ is the principal value of $\sin^{-1} a$, we know from Art. 78 that

$$\sin^{-1} a = n\pi + (-1)^n \phi.$$

Similarly for the other functions, if ϕ denotes the principal value in each case, $\tan^{-1} a = n\pi + \phi$, $\sec^{-1} a = 2n\pi \pm \phi$, etc.

In the numerical cases the principal value is usually meant.

85. If $\sin \phi = a$ (a positive) then $\cos \phi = \sqrt{1-a^2}$,

$$\tan \phi = a/\sqrt{1-a^2},\ \cot \phi = \frac{\sqrt{1-a^2}}{a},\ \sec \phi = \frac{1}{\sqrt{1-a^2}},\ \text{etc.}$$

We can thus write various inverse functions, all of which denote ϕ.

$$\phi = \sin^{-1} a = \cos^{-1} \sqrt{1-a^2} = \tan^{-1} \frac{a}{\sqrt{1-a^2}} = \cot^{-1} \frac{\sqrt{1-a^2}}{a}$$

$$= \sec^{-1} \frac{1}{\sqrt{1-a^2}} = \csc^{-1} \frac{1}{a} = \text{vers}^{-1} (1 - \sqrt{1-a^2}).$$

EXERCISE XXXVI

1. Given $\tan \phi = a$; prove $\phi = \sin^{-1} \dfrac{a}{\sqrt{1+a^2}} = \cos^{-1} \dfrac{1}{\sqrt{1+a^2}}$

$= \cot^{-1} \dfrac{1}{a} = \sec^{-1} \sqrt{1+a^2} = \csc^{-1} \dfrac{\sqrt{1+a^2}}{a}$.

Given the following functions of ϕ, write six inverse expressions in each case:

2. $\cos \phi = -a$.

3. $\cot \phi = a$.

4. $\sec \phi = -a$.

5. $\csc \phi = \pm a$.

6. $\text{vers}\, \phi = a$.

86. From the relations of earlier chapters we can derive further relations between the inverse functions. Thus, for principal values and a and b positive, if $\theta = \text{arc sin } a$ and ϕ

$= \arcsin b$, then $\sin\theta = a$, $\cos\theta = \sqrt{1-a^2}$, $\sin\phi = b$, $\cos\phi = \sqrt{1-b^2}$ and $\sin(\phi \pm \theta) = \sin\phi\cos\theta \pm \cos\phi\sin\theta$.

$$= b\sqrt{1-a^2} \pm a\sqrt{1-b^2}.$$

Therefore, writing the inverse equation

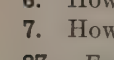
$$\sin^{-1} b \pm \sin^{-1} a = \sin^{-1}(b\sqrt{1-a^2} \pm a\sqrt{1-b^2}).$$

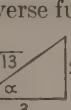
EXERCISE XXXVII

For a and b positive and for principal values, verify the following:

1. $\cos^{-1} a \pm \cos^{-1} b = \cos^{-1}(ab \mp \sqrt{1-a^2}\sqrt{1-b^2})$.
2. $\tan^{-1} a \pm \tan^{-1} b = \tan^{-1}\left(\dfrac{a \pm b}{1 \mp ab}\right)$.
3. $2\sin^{-1} a = \sin^{-1}(2a\sqrt{1-a^2})$.
4. $2\cos^{-1} a = \cos^{-1}(-1+2a^2)$.
5. $2\tan^{-1} a = \tan^{-1}\left(\dfrac{2a}{1-a^2}\right)$.
6. How is **2** changed if a and b are negative?
7. How is **4** changed if a is negative?

87. *Examples.*—A method of verifying relations between inverse functions will be illustrated by the following examples in which only principal values are used.





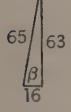


FIG. 39

1. Show that $2\cos^{-1} 3/\sqrt{13} + \cos^{-1}\frac{16}{65} + \frac{1}{2}\tan^{-1}\frac{24}{7} = \pi$.

Let $\alpha = \cos^{-1} 3/\sqrt{13}$, $\beta = \cos^{-1}\frac{16}{65}$, $\gamma = \tan^{-1}\frac{24}{7}$.

Then $\sin\alpha = 2/\sqrt{13}$, $\cos\alpha = 3/\sqrt{13}$, etc.

$\sin\beta = \frac{63}{65}$, $\cos\beta = \frac{16}{65}$, etc.

$\sin\gamma = \frac{24}{25}$, $\cos\gamma = \frac{7}{25}$, etc.

Assuming the given relation to hold and substituting in it, we have,

$$2\alpha + \beta + \tfrac{1}{2}\gamma = \pi,$$

or

$$2\alpha + \beta = \pi - \tfrac{1}{2}\gamma,$$

and, therefore, $\sin(2\alpha+\beta) = \sin\left(\pi - \dfrac{\gamma}{2}\right)$,

or, $$\sin 2\alpha \cos \beta + \cos 2\alpha \sin \beta = \sin \frac{\gamma}{2},$$

or, $$2 \sin \alpha \cos \alpha \cos \beta + (\cos^2\alpha - \sin^2\alpha) \times \sin \beta = \sqrt{\frac{1-\cos\gamma}{2}},$$

and then, $$2\times 2/\sqrt{13}\times 3/\sqrt{13}\times\frac{16}{65}+\left(\frac{9}{13}-\frac{4}{13}\right)\times\frac{63}{65}=\sqrt{\frac{1-\frac{7}{25}}{2}}.$$

$$\frac{12\times 16}{13\times 65}+\frac{5\times 63}{13\times 65}=\frac{3}{5},$$

$$\frac{192+315}{13\times 65}=\frac{3}{5}.$$

Therefore, $2\alpha+\beta+\frac{1}{2}\gamma=\pi$,

or $2\cos^{-1} 3/\sqrt{13}+\cos^{-1}\frac{16}{65}+\frac{1}{2}\tan^{-1}\frac{24}{7}=\pi$.

2. Show that $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{16}{65}=\pi/2$. Assuming the relation to be true we may write, representing the angles by inverse tangents,

$$\tan^{-1}\tfrac{4}{3}+\tan^{-1}\tfrac{5}{12}=\pi/2-\tan^{-1}\tfrac{16}{63}.$$

This equation expresses the sum of two angles as equal to the difference of two others; hence we may write

$$\tan(\tan^{-1}\tfrac{4}{3}+\tan^{-1}\tfrac{5}{12})=\tan(\pi/2-\tan^{-1}\tfrac{16}{63})=\cot(\tan^{-1}\tfrac{16}{63}),$$

$$\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\times\frac{5}{12}}=\frac{63}{16},$$

$$\frac{63}{16}=\frac{63}{16}.$$

Therefore, $$\sin^{-1}\tfrac{4}{5}+\cos^{-1}\tfrac{12}{13}+\sin^{-1}\tfrac{16}{65}=\pi/2.$$

EXERCISE XXXVIII

Show that:

1. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{5}{6}=\tan^{-1}\frac{1}{11}$.

2. $\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{5}{3}=\tan^{-1}1+\tan^{-1}\frac{8}{19}$.

3. $\sin^{-1}\frac{5}{13}=\frac{1}{2}\tan^{-1}\frac{120}{119}$.

4. $\tan^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{\sqrt{10}}=\frac{\pi}{4}$.

5. $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\cos^{-1}\frac{2}{\sqrt{5}}$.

6. $\tan^{-1}\frac{32}{43}-\tan^{-1}\frac{1}{4}=2\tan^{-1}\frac{1}{5}$.

7. $\cot^{-1}\frac{9}{2}+\tan^{-1}\frac{1}{4}=\frac{1}{2}\cos^{-1}\frac{3}{5}$.

8. $\sin\left(\tan^{-1}\frac{3}{4}\right)=\tan\left(\cos^{-1}\frac{5}{\sqrt{34}}\right)$.

9. $\cos\left(2\cos^{-1}\frac{7}{\sqrt{50}}\right)=\sin\left(4\sin^{-1}\frac{1}{\sqrt{10}}\right)$.

10. $\pi/4=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$.

11. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{13}{85}-\sin^{-1}1=0$.

12. $\sin^{-1}\frac{1}{3}=\frac{1}{3}\sin^{-1}\frac{23}{27}$.

13. $\tan^{-1}\frac{47}{52}+\cos^{-1}\frac{41}{49}=2\sin^{-1}\frac{2}{7}+3\cot^{-1}4$.

14. $\sec^{-1}\frac{13}{12}+\cot^{-1}4=\tan^{-1}\frac{32}{43}$.

15. $\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}-\cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}\right)=\frac{1}{2}$.

16. $\csc^{-1}\frac{\sqrt{73}}{3}+\sec^{-1}\frac{\sqrt{146}}{11}+\csc^{-1}2=\frac{5\pi}{12}$.

17. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}=\tan^{-1}\frac{1}{3}$.

18. $\sec^2\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)+\csc^2\left(\cos^{-1}\frac{3}{\sqrt{10}}\right)=15$.

19. Find θ if $\theta=\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{5}$.

88. Solution of Equations Involving Inverse Functions.—The following examples are illustrations of the solution of

equations involving inverse functions. We shall limit ourselves to principal values.

1. Solve $\sin^{-1} x+\sin^{-1} 2x=\frac{\pi}{3}$.

We may write $\cos(\sin^{-1}x+\sin^{-1}2x)=\cos\frac{\pi}{3}$,

or $$(\sqrt{1-x^2}\cdot\sqrt{1-4x^2}-x\cdot 2x)=\cos\frac{\pi}{3}.$$

Therefore, $\sqrt{(1-x^2)(1-4x^2)}-2x^2=\cos\frac{\pi}{3}=\frac{1}{2}$,

or $$\sqrt{(1-x^2)(1-4x^2)}=2x^2+\frac{1}{2}.$$

Squaring $1-5x^2+4x^4=4x^4+2x^2+\frac{1}{4}$.

$$7x^2=\frac{3}{4}.$$

Therefore, $x=\frac{1}{2}\sqrt{\frac{3}{7}}$.

(for $x=-\frac{1}{2}\sqrt{\frac{3}{7}}$ the principal values of $\sin^{-1}x$ and $\sin^{-1}2x$ would be in the third quadrant and hence impossible)

2. $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$.

Combining the left hand members

$$\tan^{-1}\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\times\frac{x+1}{x+2}}=\frac{\pi}{4},$$

$$\tan^{-1}\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}=\frac{\pi}{4}.$$

Therefore, $$\frac{2x^2-4}{-3}=\tan\frac{\pi}{4}=1.$$

$$2x^2-4=-3,$$

$$2x^2=1.$$

Therefore, $$x=\pm\frac{1}{\sqrt{2}}.$$

EXERCISE XXXIX

Solve for x:

1. $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$.
2. $\cos^{-1}x-\sin^{-1}x=\tan^{-1}\frac{\sqrt{3-x^2}}{x}$.
3. $2\tan^{-1}\frac{1}{2}+\cos^{-1}\frac{3}{5}=\sin^{-1}\frac{1}{x}$.
4. $\sin^{-1}\frac{8}{17}+\tan^{-1}\frac{2x}{1-x^2}=\cos^{-1}\frac{3}{5}$.
5. $\sin\left[2\cos^{-1}\left\{\tan\left(90^\circ-2\tan^{-1}x\right)\right\}\right]=0$.
6. $\csc^{-1}\frac{x}{5}+\csc^{-1}\frac{x}{12}=\frac{\pi}{2}$.
7. $\tan^{-1}2x+\tan^{-1}3x=\frac{3\pi}{4}$.
8. $\cos^{-1}\frac{x^2-1}{x^2+1}+\cot^{-1}\frac{x^2-1}{2x}=\frac{2\pi}{3}$.
9. $\tan(\sin^{-1}\sqrt{1-x^2})=\sin(\tan^{-1}2)$.
10. $\cot^{-1}\frac{x+1}{x-1}+\cot^{-1}\frac{2x+1}{2x-1}=\cot^{-1}\frac{36}{23}$.

CHAPTER X

COMPLEX NUMBERS: DE MOIVRE'S THEOREM

89. Polar Coördinates.—We shall now consider another method of locating a point in a plane by means of two real numbers. Let us have given a fixed point O, called the *pole*, and a fixed line through it OL, called the *polar axis*. Then any point in the plane, as P, determines a length $OP=r$ and an angle $LOP=\theta$. Then r and θ are the polar coördinates of P; r is called the *radius vector* and θ the *vectorial angle*. The angle θ is positive if counter-clockwise and negative if clockwise. The point P is designated by (r, θ).

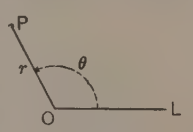


FIG. 40

To plot a point in polar coördinates lay off the vectorial angle and draw the terminal side; measure off on the terminal side a distance $OP=r$. A few illustrations are shown in Fig. (41).

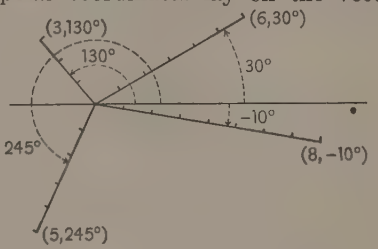


FIG. 41

EXERCISE XL

Plot the following points:

1. $(3, 200°), (6, -60°), (10, 420°), \left(8, \frac{5\pi}{4}\right), \left(16, \frac{4\pi}{3}\right), \left(50, \frac{\pi}{2}\right),$ $(10, -60°), (6, 60°), \left(4, -\frac{3\pi}{4}\right).$

90. Negative Values of the Radius Vector.—The distance r as used in rectangular coördinates was always positive; in polar coördinates, however, r may be either positive or negative. If P lies on the terminal side of θ, r is positive; *r is negative if P lies on the terminal side produced through the pole.* A few illustrations are shown in Fig. 42.

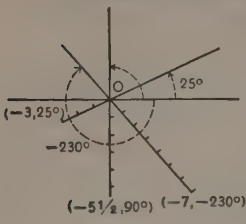


FIG. 42

EXERCISE XLI

2. Plot the following points:

$$(-5,\ 70°),\ \left(-2.1,\ \frac{7\pi}{3}\right),\ (-3.7,\ 26°),\ (-12,\ -175°),\ (-4,\ 620°),\ (-7,\ -431°).$$

91. Given $(r,\ \theta)$ there is but one point in the plane which has these values for its polar coördinates. On the other hand we can write the polar coördinates of any given point in an infinite number of ways, for $(r,\ \theta)$, $(r,\ 2\pi+\theta)$, $(r,\ \theta-2\pi)\ \ldots$ $(r,\ 2n\pi+\theta)$, $(-r,\ \pi+\theta)$, $(-r,\ (2n+1)\pi+\theta)$ all represent the same point. For example, the point (6, 30°), Fig. 41, is also represented by (6, −330°), (−6, 210°), (−6, −150°), etc.

92. Transformation from Rectangular to Polar Coördinates.—Let us have given a point P located by its rectangular coördinates $(x,\ y)$. Draw OP; then if we consider O as the pole in the polar system, $OP=r$ will be the radius vector and $\angle XOP=\theta$ will be the vectorial angle.

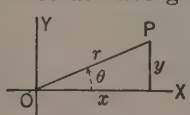


FIG. 43

From Fig. 43 we see that

$$\mathbf{x=r\cos\theta,\quad y=r\sin\theta,\quad r=\sqrt{x^2+y^2},}$$

$$\mathbf{\theta=\tan^{-1}\frac{y}{x}=\cos^{-1}\frac{x}{\sqrt{x^2+y^2}}=\sin^{-1}\frac{y}{\sqrt{x^2+y^2}}.}$$

These equations allow us to pass from polar to rectangular coördinates or conversely.

Example 1 —Given the polar coördinates of a point as $\left(-3, \frac{\pi}{6}\right)$ what are its rectangular coördinates?

$$x = r \cos\theta = -3 \cos 30° = -\frac{3\sqrt{3}}{2}.$$

$$y = r \sin\theta = -3 \sin 30° = -\frac{3}{2}.$$

Example 2. Given the rectangular coördinates of a point as $(4, -3)$ what are its polar coördinates?

$$r = \sqrt{x^2+y^2} = \sqrt{16+9} = 5.$$

$$\theta = \tan^{-1} -\frac{3}{4} = -36°52'12''$$

Hence polar coördinates are $(5, -36°52'12'')$. However, this could be written as $(5, 323°7'48'')$, $(-5, 143°7'48'')$, $(-5, -216°52'12'')$ etc.

EXERCISE XLII

1. Transform to polar coördinates, getting two pairs of values for r and θ for each point

$(1, \sqrt{3})$, $(-1, -\sqrt{3})$, $(2, -2)$, $(-2, 2)$, $(5, 12)$, $(-3, 4)$.

2. Transform to rectangular coördinates:

$\left(3, \frac{\pi}{3}\right)$, $(4, -213°)$, $(7, 113°)$, $(-7, 300°)$, $(27, 825°)$.

3. What points are represented by $r = 10$? By $\theta = 30°$?

93. Complex Numbers.—From algebra we are familiar with equations of the form $x^2+4=0$. The roots of this equation are $\pm\sqrt{-4}$. Such a number as this (i.e., the even root of a negative number) is called an "imaginary" number and is usually written as $\pm 2\sqrt{-1}$ or $\pm 2j$ where $j = \sqrt{-1}$.

Again, if we solve an equation such as $x^2+2x+2=0$ we obtain for roots $\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j$. A number such as $-1+j$ which is the sum or difference of a real and an imaginary number is called a complex number. The general form of a complex number is $x+yj$ where x and y are any two real numbers.

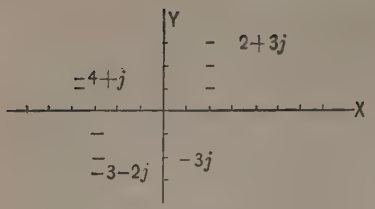


FIG. 44

We have seen that real numbers may be represented by points on a straight line. We can represent a complex number by a point in a plane, the abscissa of the point being x, the real part of the number and the ordinate being y, the coefficient of j. Briefly, $x+yj$ is represented by the point (x, y) in a system of rectangular coördinates. Figure 44 gives several illustrations.

94. Polar Form of Complex Numbers.—It is often more convenient to transform the complex number $x+yj$ to polar coördinates. Thus,

$$x+yj=r\cos\theta+jr\sin\theta$$
$$=r(\cos\theta+j\sin\theta),$$

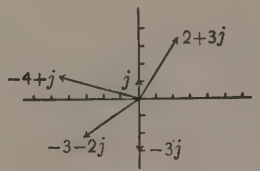


FIG. 45

where r is called the modulus or absolute value, and θ is called the argument or amplitude of the complex number.

We have a second graphical representation for a complex number, as we may let the vector joining the origin to the point (x, y) represent it. Thus, we have Fig. 45 representing the numbers shown in Fig. 44.

$2+3j=\sqrt{13}\,(\cos 56°19'+j\sin 56°19')$.
$j=1\,(\cos 90°+j\sin 90°)$.
$-4+j=\sqrt{17}\,(\cos 165°58'+j\sin 165°58')$.
$-3-2j=\sqrt{13}\,(\cos 213°41'+j\sin 213°41')$.
$-3j=3\,(\cos 270°+j\sin 270°)$.

It is well to note that $r(\cos\theta+j\sin\theta)$ is not the only polar expression for $x+yj$. By referring to Art. 91 it can be seen that any of the forms, $r[\cos(2n\pi+\theta)+j\sin(2n\pi+\theta)]$, gives a polar expression for $x+yj$.

95. Complex Numbers Equal.—If $x+yj$ and $a+bj$ are any two complex numbers, they are said to be equal if $x=a$ and

$y=b$. Hence, if *two complex numbers are equal the real parts are equal each to each and the imaginary parts are equal each to each.*

96. Vector Notation.—A vector is often represented by placing a dot under its symbol. Thus $\dot{A}=x+yj$. If r is the modulus and θ the amplitude, for the first quadrant we shall write $\dot{A}=r\angle\theta$, for the second quadrant $\dot{A}=r\angle\theta$, for the third quadrant $\dot{A}=r\angle\theta$ or $\dot{A}=r\angle-\theta$, and for the fourth quadrant $\dot{A}=r\angle\theta$ or $\dot{A}=r\angle-\theta$. For example, the vectors of Fig. 45 can be denoted:

$$\dot{A}=2+3j=\sqrt{13}\angle 56°19',$$
$$\dot{B}=j=1\angle 90°,$$
$$\dot{C}=-4+j=\sqrt{17}\angle 165°58',$$
$$\dot{D}=-3-2j=\sqrt{13}\angle 146°19' \text{ or } \sqrt{13}\angle -146°19',$$
$$\dot{E}=-3j=3\angle 90° \text{ or } 3\angle -90°.$$

This notation will be used in work with alternating currents.

EXERCISE XLIII

Express each of the following in the polar form. Find the amplitude and modulus and plot:

1. $-2-3j$.
2. $5+4j$.
3. $1-j$.
4. $\frac{4}{5}+\frac{3}{2}j$.
5. $18.9+1.36j$.

Write the following in the form $x+yj$:

6. $5(\cos 60°-j \sin 60°)$.
7. $2(\cos 135°+j \sin 135°)$.
8. $4(\cos 270°+j \sin 270°)$.
9. $5(\cos 75°+j \sin 75°)$.
10. $(\cos 22\frac{1}{2}°+j \sin 22\frac{1}{2}°)$.
11. $3(\cos 43°+j \sin 43°)$.

12. $(\cos 210° + j \sin 210°)$.

13. Simplify j, j^2, j^3, j^4, j^5, etc.
Simplify $j^{17}+j^{23}+j^{19}+j^{33}+j^{14}+j^{32}$.

Solve the following equations for real values of x and y:

14. $x+y+j(x-y)=2+8j$.

15. $2x+5y+j(3x-2y)=-4-j$.

16. $2x^2+3y^2=25-(3x+4y-36)j$.

97. Addition and Subtraction.—By assuming that complex numbers obey the laws of algebra, we may write for any two complex numbers,

$$A=x+yj \quad \text{and} \quad B=a+bj,$$

$$A+B=(x+yj)+(a+bj)=(x+a)+(y+b)j,$$

and

$$A-B=(x+yj)-(a+bj)=(x-a)+(y-b)j.$$

Thus to add or subtract two complex numbers we perform the operation with the real parts and with the imaginary parts separately. In Fig. 46 the vectors representing the complex numbers A, B and the sum $A+B$ are plotted. From the figure we see that to add two complex numbers, we construct at the extremity of the vector representing one of them a vector equal in magnitude and parallel to the second, and the sum is the vector drawn from the origin to the extremity of the second vector. If we complete the parallelogram of which the two given vectors are adjacent sides, then the sum is the vector equal to the diagonal drawn from the origin.

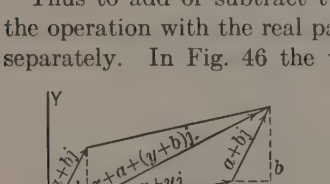


FIG. 46

To perform subtraction $(A-B)$ we can add the vectors $(x+yj)$ and $(-a-bj)$.

EXERCISE XLIV

Perform the indicated operations graphically and algebraically.

1. $(3+2j)+(4+5j)$.

2. $(3-2j)-(1+j)$.

3. $(0+4j)+(7+2j)$.

4. $(37+43j)-(23+5j)$.

5. $(2+3j)+(7+4j)-(3-2j)$.

6. If $a+bj$ is added to and subtracted from $x+yj$, show that the vector representing the sum is the longer diagonal, and the vector representing the difference is the shorter diagonal, of the parallelogram of which the two given vectors are adjacent sides.

7. Show that the amplitude of the sum of three complex numbers is the amplitude of the vector forming a quadrilateral with the three vectors representing the three given complex numbers. Generalize this statement for any number of vectors.

98. Product and Quotient of Two Complex Numbers.— Given two complex numbers $\dot{A}=r_1\angle\theta_1 = r_1(\cos\theta_1+j\sin\theta_1)$ and $\dot{B}=r_2\angle\theta_2=r_2(\cos\theta_2+j\sin\theta_2)$, on assuming the laws of algebra to hold, we have for the product

$$\begin{aligned}\dot{A}\dot{B}&=r_1r_2\angle\theta_1\ \angle\theta_2=r_1r_2(\cos\theta_1+j\sin\theta_1)(\cos\theta_2+j\sin\theta_2)\\&=r_1r_2(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+j\sin\theta_1\cos\theta_2+j\cos\theta_1\sin\theta_2)\\&=r_1r_2[\cos(\theta_1+\theta_2)+j\sin(\theta_1+\theta_2)]=r_1r_2\angle\theta_1+\theta_2.\end{aligned}$$

Thus: *The modulus of the product of two complex numbers is the product of their moduli, and the amplitude is the sum of their amplitudes.*

In particular when a complex number is multiplied by j the vector representing the number is rotated through an angle of 90°. For example, referring to Art. 27 where numbers are represented by points on a scale the point 2 is located at 2 units to the right; when 2 is multiplied by -1 it becomes -2 and is

located 2 units to the left; in other words the vector representing 2 is revolved through 180°. But the multiplication by -1 represents two successive multiplications by j. Hence we would expect a single multiplication by j to mean a rotation through 90°. That this is true can be illustrated by performing the multiplication of 2 by j by means of the theorem just proved.

$$2=2(\cos 0°+j \sin 0°)=2\underline{/0°};$$

$$j=1(\cos 90°+j \sin 90°)=1\underline{/90°};$$

$$2\cdot j=2[\cos (0°+90°)+j \sin (0°+90°)]=2(\cos 90°+j \sin 90°)=2\underline{/90°}.$$

Similarly, for the quotient

$$A/B=\frac{r_1 (\cos \theta_1+j \sin \theta_1)}{r_2 (\cos \theta_2+j \sin \theta_2)}=\frac{r_1 (\cos \theta_1+j \sin \theta_1)(\cos \theta_2-j \sin \theta_2)}{r_2 (\cos \theta_2+j \sin \theta_2)(\cos \theta_2-j \sin \theta_2)},$$

$$=\frac{r_1}{r_2}\cdot\frac{(\cos \theta_1 \cos \theta_2+\sin \theta_1 \sin \theta_2)+j(\sin \theta_1 \cos \theta_2-\cos \theta_1 \sin \theta_2)}{(\cos^2 \theta_2+\sin^2 \theta_2)},$$

$$=\frac{r_1}{r_2}[\cos (\theta_1-\theta_2)+j \sin (\theta_1-\theta_2)]=\frac{r_1}{r_2}\underline{/\theta_1-\theta_2}.$$

Thus: *The modulus of the quotient of two complex numbers is the quotient of their moduli, and the amplitude is the difference of their amplitudes.*

99. De Moivre's Theorem.—To find the product of n complex numbers we can apply the rule for multiplication n times in succession—each time multiplying the product already obtained by the next factor. Thus

$$r_1 (\cos \theta_1+j \sin \theta_1)\cdot r_2(\cos \theta_2+j \sin \theta_2) \cdots r_n (\cos \theta_n+j \sin \theta_n)$$

$$=r_1 r_2 (\cos_1 \theta_1+\theta_2 +j \sin \theta_1+\theta_2)\cdot r_3 (\cos \theta_3+j \sin \theta_3) \cdots r_n (\cos \theta_n +j \sin \theta_n),$$

$$=r_1 r_2 r_3 (\cos \theta_1+\theta_2+\theta_2+j \sin \theta_1+\theta_2+\theta_3)\cdot r_4 (\cos \theta_4+j \sin \theta_4)\cdots r_n (\cos \theta_n+j \sin \theta_n),$$

$$=r_1 r_2 \cdots r_n [\cos (\theta_1+\theta_2+\theta_3+\cdots+ \theta_n)+j \sin (\theta_1+\theta_2+\theta_3 +\cdots+\theta_n)].$$

If all of the factors are equal, we have

(A′) $r^n(\cos\theta+j\sin\theta)^n=r^n(\cos n\theta+j\sin n\theta)$ or $\underline{[r/\theta]}^n=\underline{r^n/\theta n}$.

(A) $\mathbf{(\cos\theta+j\sin\theta)^n=(\cos n\theta+j\sin n\theta)}$ **or** $\underline{(/\theta)^n=/\theta n}$.

The relation indicated by this equation is known as De Moivre's Theorem. It has been derived for n a positive integer but we shall now prove it for any rational exponent n. We have

$$(\cos\theta+j\sin\theta)^{-1}=\frac{1}{\cos\theta+j\sin\theta}=\frac{(\cos 0°+j\sin 0°)}{\cos\theta+j\sin\theta}$$
$$=\cos(0°-\theta)+j\sin(0°-\theta),$$
$$=\cos(-\theta)+j\sin(-\theta).$$

Thus

$$(\cos\theta+j\sin\theta)^{-n}=[(\cos\theta+j\sin\theta)^{-1}]^n$$
$$=[\cos(-\theta)+j\sin(-\theta)]^n=\cos(-n\theta)+j\sin(-n\theta);$$

or the relation holds for negative integers as well.

If p is any integer and $\theta=p\phi$,

$$[\cos\theta+j\sin\theta)^{\frac{1}{p}}=[\cos p\phi+j\sin p\phi]^{\frac{1}{p}}=[(\cos\phi+j\sin\phi)^p]^{\frac{1}{p}},$$
$$=\cos\frac{\theta}{p}+j\sin\frac{\theta}{p}.$$

Again for any integers n and p prime to each other

$$[\cos\theta+j\sin\theta]^{\frac{n}{p}}=\left(\cos\frac{\theta}{p}+j\sin\frac{\theta}{p}\right)^n=\cos\frac{n\theta}{p}+j\sin\frac{n\theta}{p}.$$

Thus the relation (A) holds for all rational values of n.

100. *Examples.*—Change to polar form and

1. Find the product of $-\frac{1}{2}-\frac{1}{2}\sqrt{3}\,j$ and $4+4j$.

To put the first number in the polar form we note $x=-\frac{1}{2}$, $y=-\frac{1}{2}\sqrt{3}$. Then $\sqrt{x^2+y^2}=1$, $\cos\theta=-\frac{1}{2}$, $\sin\theta=-\frac{1}{2}\sqrt{3}$, $\tan\theta=+\sqrt{3}$ and $\theta=240°$ or $2n\pi+\frac{4\pi}{3}$, and

$$-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}=1\,(\cos 240°+j\sin 240°)=1/\underline{120°}.$$

For the second number $x=4$, $y=4$. Then $r=\sqrt{16+16}=4\sqrt{2}$, $\theta=45°$ or $2n\pi+\frac{\pi}{4}$. Thus,

$$4+4j=4\sqrt{2}\,(\cos 45°+j\sin 45°)=4\sqrt{2}\angle 45°.$$

Therefore, $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}j\right)(4+4j)=1\angle 120°\cdot 4\sqrt{2}\angle 45°$,

$$=4\sqrt{2}\angle 45°-120°,$$
$$=4\sqrt{2}\angle -75°,$$
$$=4\sqrt{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}-j\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$
$$=2[(\sqrt{3}-1)-j(\sqrt{3}+1)].$$

Verify this result by multiplying the given numbers in their algebraic form.

2. Find the value of $4+4j$ divided by $\frac{\sqrt{3}}{2}-\frac{1}{2}j$.

Solution.—$4+4j=4\sqrt{2}\angle 45°$, $\frac{\sqrt{3}}{2}-\frac{1}{2}j=1\angle 30°$.

Then $$\frac{4+4j}{\frac{\sqrt{3}}{2}-\frac{1}{2}j}=\frac{4\sqrt{2}\angle 45°}{1\angle 30°},$$

$$=4\sqrt{2}\angle 45°-(-30°),$$
$$=4\sqrt{2}\angle 75°.$$

3. Find $(2+2j)^{10}$.

In polar form $2+2j=2\sqrt{2}\angle 45°$.

Then $(2+2j)^{10}=(2\sqrt{2}\angle 45°)^{10}$,

$$=(2\sqrt{2})^{10}\angle 450°,$$
$$=2^{15}\angle 90°.$$

EXERCISE XLV

Perform the operations indicated with the following complex numbers changing first to the polar form. Plot each number and the result:

1. $(1+j)\times(-\frac{1}{2}-\frac{1}{2}\sqrt{3}j)$.
2. $(3+3j)\times(7+7j)$.
3. $(1+j)\times(2-2j)$.
4. $(1-j)^2\div(2+2\sqrt{3}j)$.
5. $\dfrac{(-2+3j)(8-5j)}{7+9j}$.
6. If $z=(\cos\theta+j\sin\theta)$ show that $\dfrac{1}{z}=\cos\theta-j\sin\theta$.

Find the value of each of the following:

7. $(\sqrt{3}+j)^4$.
8. $(1-3j)^8$.
9. $(1+j)^7$.
10. $(2+5j)^4+(2-5j)^4$.

101. Roots of a Complex Number.—We shall first show that every complex number (not zero) has just n nth roots.

Reducing any complex number to the trigonometric form we have $x+yj=r\,[\cos(2k\pi+\theta)+j\sin(2k\pi+\theta)]$ (Art. 96.) By De Moivre's Theorem (A′)

$$\sqrt[n]{A}=(x+yj)^{\frac{1}{n}}=r^{\frac{1}{n}}[\cos(2k\pi+\theta)+j\sin(2k\pi+\theta)]^{\frac{1}{n}}$$

$$=r^{\frac{1}{n}}\left[\cos\frac{2k\pi+\theta}{n}+j\sin\frac{2k\pi+\theta}{n}\right]$$

$$=\sqrt[n]{r}\,\underline{/\frac{2k\pi+\theta}{n}}.$$

By putting $k=0, 1, 2, \cdots n-1$ successively, we get n different complex numbers all of which when raised to the nth power are equal to the given number.

Therefore: Every complex number (not zero) has just n different nth roots.

Example.—Find the fifth roots of $-\sqrt{3}+3j=z^5$.

Here $r=\sqrt{9+3}=2\sqrt{3}$, $\theta=\tan^{-1}\dfrac{3}{-\sqrt{3}}=\tan^{-1}-\sqrt{3}$.

$z^5=-\sqrt{3}+3j=2\sqrt{3}\,[\cos(k\cdot 360°+120°)+j\sin(k\cdot 360°+120°)]$.

Then
$$z=(\sqrt{3}+3j)^{\frac{1}{5}}$$
$$=(2\sqrt{3})^{\frac{1}{5}}\left[\cos\frac{k\cdot 360°+120°}{5}+j\sin\frac{k\cdot 360°+120°}{5}\right].$$
$$=(2\sqrt{3})^{\frac{1}{5}}\angle k\cdot 72°+24°.$$

For $k=0$, $z_1=(2\sqrt{3})^{\frac{1}{5}}\angle 24°$.

For $k=1$, $z_2=(2\sqrt{3})^{\frac{1}{5}}\angle 96°$.

For $k=2$, $z_3=(2\sqrt{3})^{\frac{1}{5}}\angle 168°$.

For $k=3$, $z_4=(2\sqrt{3})^{\frac{1}{5}}\angle 120°$.

For $k=4$, $z_5=(2\sqrt{3})^{\frac{1}{5}}\angle 48°$.

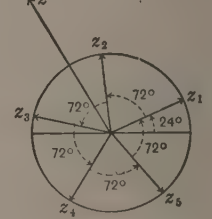


FIG. 47

The roots are represented graphically in Fig. 47.

EXERCISE XLVI

1. Find $\sqrt[3]{1+j}$.
2. Find $\sqrt[5]{\frac{1}{2}-\frac{1}{2}\sqrt{3}j}$.
3. Find $\sqrt{9-8j}$.
4. Find $\sqrt[7]{3-3j}$

Solve the equations:

5. $x^3=1$.
6. $x^3+1=0$.
7. $x^5=32$.
8. $x^4=1$.
9. $x^6=64$.
10. $x^8+1=0$.

102. Applications of De Moivre's Theorem to Trigonometry.

1. The following example will indicate a method for finding formulae expressing the sin $n\theta$ and cos $n\theta$ in terms of the powers of sin θ and cos θ.

Example. Find formulae for $\sin 3\theta$ and $\cos 3\theta$ in powers of $\sin\theta$ and $\cos\theta$. By De Moivre's Theorem

$$(\cos 3\theta + j \sin 3\theta) = (\cos\theta + j \sin\theta)^3,$$

expanding by the Binomial Theorem,

$$= \cos^3\theta + 3\cos^2\theta\,(j\sin\theta) + 3\cos\theta\,(j\sin\theta)^2 + (j\sin\theta)^3,$$

reducing the various powers of j,

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + 3j\sin\theta\cos^2\theta - j\sin^3\theta.$$

Equating the real and imaginary parts (see Art. 95),

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$
$$\sin 3\theta = -\sin^3\theta + 3\sin\theta\cos^2\theta.$$

2. We can find expressions for $\sin^n\theta$ and $\cos^n\theta$ in terms of functions of multiples of θ in the following manner.

$$\left.\begin{aligned} \text{Let } z &= \cos\theta + j\sin\theta, \\ \text{then } \frac{1}{z} &= \cos\theta - j\sin\theta, \end{aligned}\right\} \qquad (a)$$

$$z^n = \cos n\theta + j\sin n\theta, \quad \frac{1}{z^n} = \cos n\theta - j\sin n\theta. \qquad (b)$$

Adding and subtracting the first pair of equations (a) we have

$$z + \frac{1}{z} = 2\cos\theta, \quad z - \frac{1}{z} = 2j\sin\theta, \qquad (c)$$

and again the second pair (b)

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \quad z^n - \frac{1}{z^n} = 2j\sin n\theta. \qquad (d)$$

Then raising $2\cos\theta = z + \frac{1}{z}$ to the nth power and expanding

$$2^n\cdot\cos^n\theta = \left(z + \frac{1}{z}\right)^n = z^n + nz^{n-2} + \frac{n(n-1)}{1\cdot 2}\cdot z^{n-4} + \cdots \text{etc.}$$

Since the binomial coefficients equally distant from the ends are equal,

$$2^n \cos^n \theta = \left(z^n + \frac{1}{z^n}\right) + n\left(z^{n-2} + \frac{1}{z^{n-2}}\right) + \frac{n(n-1)}{1 \cdot 2}\left(z^{n-4} + \frac{1}{z^{n-4}}\right)$$
$$+ \cdots \text{ etc.}$$

Substituting from the relations (d)

$$\cos^n \theta = \frac{1}{2^n}[2 \cos n\theta + n(2 \cos(n-2)\theta) + \frac{n(n-1)}{1 \cdot 2} \cdot (2 \cos(n-4)\theta)$$
$$+ \cdots \text{ etc.}]$$

In a similar way an expression for $\sin^n \theta$ can be obtained as is shown by the example. In this case care must be taken with regard to signs and powers of j.

$\sin^7 \theta$

$$= \left(\frac{1}{2j}\right)^7 \left(z - \frac{1}{z}\right)^7 = \frac{1}{(2j)^7}\left(z^7 - 7z^5 + 21z^3 - 35z + 35\frac{1}{z} - 21\frac{1}{z^3} + 7\frac{1}{z^5} - \frac{1}{z^7}\right),$$

$$= \left(\frac{1}{2j}\right)^7 \left[\left(z^7 - \frac{1}{z^7}\right) - 7\left(z^5 - \frac{1}{z^5}\right) + 21\left(z^3 - \frac{1}{z^3}\right) - 35\left(z - \frac{1}{z}\right)\right],$$

$$= \left(\frac{1}{2j}\right)^7 [2j \sin 7\theta - 7 \cdot 2j \sin 5\theta + 21 \cdot 2j \sin 3\theta - 35 \cdot 2j \sin \theta],$$

$$= -\frac{1}{64}(\sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \sin \theta).$$

EXERCISE XLVII

Find formulae for the following:

1. $\sin 2\theta$, $\cos 2\theta$.
2. $\sin 4\theta$, $\cos 4\theta$.
3. $\sin 5\theta$, $\cos 5\theta$.
4. $\sin 6\theta$, $\cos 6\theta$.
5. $\sin^3 \theta$.
6. $\cos^3 \theta$.
7. $\sin^4 \theta$.
8. $\cos^4 \theta$.
9. $\sin^5 \theta$.
10. $\cos^5 \theta$.
11. $\sin^6 \theta$.
12. $\cos^6 \theta$.

103. Applications of Complex Quantities to Alternating Currents.—Complex numbers are extensively used in the study of alternating currents. Some examples will be mentioned.

In a simple series circuit consisting of a non-inductive resistance R and an inductive reactance X_L then the impedance E is expressed as a complex quantity:

$$E = IR + jIX_L.$$

From this the current is found to be

$$I = E\left[\frac{R}{R^2+X_L{}^2} - j\frac{X_L}{R^2+X_L{}^2}\right] = E(g-jb),$$

g is called conductance and b susceptance.

If $E = 100 - j132.6 = 166\angle 53$ volts, then $I = 0.435 + j0.577 = 0.723\angle 53°$ amp. The student should derive the equation for I from that given for E.

If $E = e_1 + je_2 = E\angle\alpha$ gives the voltage, and $I = i_1 - ji_2 = I\angle\beta$, the current of a circuit, then the power is given by the relation

$$P_1 = e_1 i_1 \quad \text{or} \quad P = EI\cos(\alpha - \beta).$$

PART II—SPHERICAL TRIGONOMETRY

CHAPTER XI

THE SPHERICAL TRIANGLE

104. The following definitions and theorems relating to spherical triangles are quoted from Spherical Geometry.

1. A *great circle* of a sphere is a section made by a plane which passes through the center of the sphere.

2. A *small circle* of a sphere is a section made by a plane which does not pass through the center of the sphere.

3. The *poles* of a circle are the ends of the diameter of the sphere which is perpendicular to the plane of the circle. The distance along a great circle arc from the circle to its nearer pole is called the polar distance of the circle.

4. The *distance* between two points on the surface of a sphere is the *arc* of the great circle which joins them. Through any two points, unless they be at the extremities of a diameter, only one great circle can be drawn.

5. The *angle* between two curves passing through the same point is the angle formed by the tangents to the two curves at that point. The angle formed by the intersection of two arcs of great circles is called a spherical angle. A spherical angle has the same measure as the dihedral angle formed by the planes of the two great circles.

6. A *lune* is a part of the surface of a sphere bounded by two semicircumferences of great circles. The angle of a lune is the angle formed by the semicircumferences.

7. A *spherical triangle* is a part of the surface of a sphere bounded by three arcs of great circles. A trihedral angle, with vertex at the center of the sphere, is formed by the planes of the

great circles which form the sides of the triangle. Spherical triangles are right, obtuse or acute, equilateral, isosceles or scalene as are plane triangles; however, a spherical triangle may

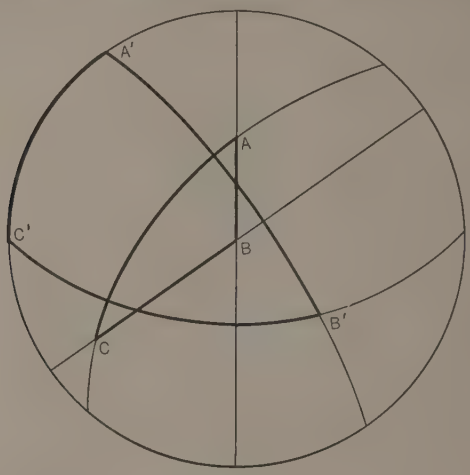


FIG. 48

have one, two, or three right or obtuse angles. We shall denote the angles by A, B, C, and the respective opposite sides by a, b, c.

THEOREMS

I. *Each side of a spherical triangle is less than the sum of the other two sides;* that is, $a<b+c$, etc.

II. *Any angle of a spherical triangle is less than* 180°; that is, $A<180°$, etc.

III. *Any side of a spherical triangle is less than* 180°; that is, $a<180°$, etc.

IV. *The sum of the sides of a spherical triangle is less than* 360°; that is, $a+b+c<360°$.

V. *The sum of the angles of a spherical triangle is greater than* 180° *and less than* 540°; that is, $180° < A+B+C < 540°$.

VI. *In any spherical triangle the order of magnitude of the angles is the same as that of the respective opposite sides; equal angles are opposite equal sides.* Thus, if $a<b<c$ then $A<B<C$.

VII. *The area of a spherical triangle is* $\pi R^2 E/180°$ *where* R *is the radius of the sphere and* $E=A+B+C-180°$. (E is called the Spherical Excess.)

8. If from the vertices of a spherical triangle as poles, arcs of great circles are drawn, they form a second triangle called the polar of the first triangle. We shall denote the parts of the polar triangle by A', B', C', a', b', c', and choose the polar triangle from the eight formed by the great circles so that A and A' are on the same side of BC, B and B' on the same side of AC and C and C' on the same side of AB. (See Fig. 48.)

VIII. *If* $A'B'C'$ *is the polar triangle of* ABC, *then reciprocally,* ABC *is the polar triangle of* $A'B'C'$.

IX. *In two polar triangles the angle of the one is the supplement of the opposite side of the other;* that is, $A+a'=180°$, $a+A'=180°$, etc.

105. The Cosine Law for Spherical Triangles.—Let $O-ABC$ in Fig. 49 denote a spherical triangle and its related trihedral angle. At A draw a tangent plane cutting the faces of the trihedral angle in AM, AN, and MN. Therefore, $\measuredangle MOA$ and NOA are right $\measuredangle$, and $\angle MAN$ measures the dihedral $BAC-O$ and hence the spherical angle $BAC=\angle A$.

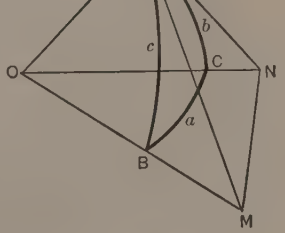


FIG. 49

From $\triangle MON$

$$\overline{MN}^2=\overline{OM}^2+\overline{ON}^2-2\ OM\cdot ON\cdot\cos\ a\ (\cos\ a=\cos\ \angle MON);$$

from $\triangle MAN$

$\overline{MN}^2 = \overline{AM}^2 + \overline{AN}^2 - 2\,AM \cdot AN \cdot \cos A \quad (\cos A = \cos \angle MAN).$

Subtracting the second equation from the first, we obtain

$0 = \overline{OM}^2 - \overline{AM}^2 + \overline{ON}^2 - \overline{AN}^2 - 2\,OM \cdot ON \cdot \cos a + 2\,AM \cdot AN \cdot \cos A,$

or $\quad 0 = 2\,\overline{OA}^2 - 2\,OM \cdot ON \cos a + 2\,AM \cdot AN \cdot \cos A;$

dividing by $2\,OM \cdot ON$, we have

$$0 = \frac{OA}{OM} \cdot \frac{OA}{ON} - \cos a + \frac{AM}{OM} \cdot \frac{AN}{ON} \cos A. \qquad (1)$$

From the figure $\frac{AM}{OM} = \sin c$, $\frac{OA}{OM} = \cos c$, $\frac{AN}{ON} = \sin b$ and $\frac{OA}{ON} = \cos b$; substituting these values in (1), we have

$$\textbf{cos a} = \textbf{cos b} \cdot \textbf{cos c} + \textbf{sin b} \cdot \textbf{sin c} \cdot \textbf{cos A}. \qquad (45)$$

Similarly,

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B,$$
$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C.$$

In the construction of Fig. 49 it is assumed that b and c are less than 90°, for the plane AMN is assumed to cut OB and OC, but the formula (45) is true when these sides are not less than 90° as can be shown with the use of proper figures.

106. If ABC and $A'B'C'$ are two polar triangles, we have $\cos A' = \cos (180° - a) = -\cos a$, $\sin a' = \sin (180° - A) = \sin A$, $\cos a' = \cos (180° - A) = -\cos A$,
and similar relations for the other letters.

By (45) we have

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A';$$

substituting for the functions of the primed letters the equivalent functions of the parts of the polar triangle, and dividing by -1, we have

$$\textbf{cos A} = -\textbf{cos B} \cdot \textbf{cos C} + \textbf{sin B} \cdot \textbf{sin C} \cdot \textbf{cos a}, \qquad (46)$$

and similarly for the other angles,

$$\cos B = -\cos A \cdot \cos C + \sin A \cdot \sin C \cdot \cos b,$$
$$\cos C = -\cos A \cdot \cos B + \sin A \cdot \sin B \cdot \cos c.$$

107. We can transform (45) into another useful form. By (20) $\cos A = 1 - 2 \text{ hav } A$, and substituting for $\cos A$, we obtain

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c - 2 \sin b \cdot \sin c \cdot \text{hav } A,$$
$$= \cos (b-c) - 2 \sin b \cdot \sin c \cdot \text{hav } A; \qquad \text{(By (13))}$$

substituting $1 - 2\text{hav}a$ for $\cos a$, and $1 - 2\text{hav}(b-c)$ for $\cos (b-c)$ we have

$$1 - 2 \text{ hav } a = 1 - 2 \text{ hav } (b-c) - 2 \sin b \cdot \sin c \cdot \text{hav } A,$$

or

$$\textbf{hav a} = \textbf{hav (b} - \textbf{c)} + \textbf{sin b} \cdot \textbf{sin c} \cdot \textbf{hav A.} \qquad (47)$$

108. The Sine Law for Spherical Triangles.—In Fig. 50, pass a plane through C perpendicular to OB. Thus, the lines RC and RT, in which it intersects the faces of the trihedral, are $\perp$ to OB and $\angle CRT = \angle B$.

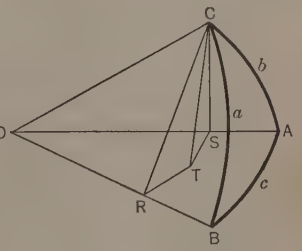


FIG. 50

In a similar way pass the plane $CST \perp$ to OA. Then SC and ST are $\perp$ to OA and $\angle CST = \angle A$.

Then, $\sin A = \sin \angle CST = \dfrac{CT}{SC}$ and $\sin B = \sin \angle CRT = \dfrac{CT}{RC}$.

Also $\sin a = \sin \angle COB = \dfrac{RC}{OC}$ and $\sin b = \sin \angle COA = \dfrac{SC}{OC}$.

Dividing, $$\frac{\sin A}{\sin B} = \frac{RC}{SC} = \frac{\sin a}{\sin b}.$$

If we drop the perpendicular from A we can obtain the relation

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}.$$

Therefore,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{48}$$

This equality indicates that: *In a spherical triangle the sines of the angles are proportional to the sines of the opposite sides.*

EXERCISE XLVIII

1. Derive the Sine formula algebraically from the Law of Cosines. Compute $\sin^2 A$, reducing the numerator to a form involving cosines only. Thus show that $\frac{\sin A}{\sin a}$ does not change when the letters are permuted.

2. Show that

$$\text{hav } (180° - A) = \text{hav } (B + C) - \sin B \cdot \sin C \cdot \text{hav } a.$$

3. Given $b = 47° 44' 00''$, $c = 53° 19' 28''$, $A = 52° 30' 00''$; find a by the Cosine formula.

4. Given $b = 118° 42' 00''$, $a = 68° 34' 25''$, $C = 41° 26' 37''$; find c by the Cosine formula.

5. Use the Haversine formula to solve **3.** (See Art. 120 for form.)

6. Given $a = 75° 18' 00''$, $c = 63° 35' 41''$, $B = 68° 22' 00''$; find b by the Cosine formula and by the Haversine formula.

7. Given $B = 85° 37' 51''$, $C = 67° 24' 41''$, $a = 64° 23' 24''$; find A by formula (46).

8. Given $B = 65° 16' 25''$, $b = 47° 44' 00''$, $C = 79° 52' 30''$; find c by the Sine Law.

9. Given $a = 40° 05' 26''$, $A = 29° 42' 34''$, $c = 26° 21' 18''$; find C.

10. Given $b = 47° 15' 00''$, $c = 52° 35' 16''$, $A = 56° 30' 00''$, find a by the Cosine formula and B and C by the Sine formula.

CHAPTER XII

THE RIGHT SPHERICAL TRIANGLE

109. Napier's Rules of Circular Parts.—The formulae necessary for the solution of a right spherical triangle can be readily obtained from the following rules given by Napier: In a right spherical triangle,

1. The sine of a middle part is equal to the product of the cosines of the opposite parts.

2. The sine of a middle part is equal to the product of the tangents of the adjacent parts. (49)

The parts mentioned in the rules are the five so-called *circular parts* of the right triangle; namely, the *complement* of the

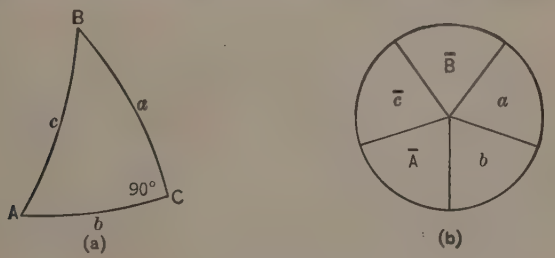


FIG. 51

hypotenuse, the *complements* of the two *oblique angles*, and the two *legs*. Thus, the circular parts of the triangle ABC (Fig. 51 (a)) are $90°-c$ (denoted $\bar{c}$), $90°-A$ (denoted $\bar{A}$), $90°-B$ (denoted $\bar{B}$), a and b. As an aid in the use of Napier's Rules a circular arrangement of these circular parts can be made as shown in Fig. 51 (b).

Any one of the five may be taken as a *middle* part, then the two adjacent to it (contiguous) are the *adjacent* parts and the other two (non-contiguous) are the *opposite* parts. If any three circular parts are selected, one is a middle part and the remaining two are either adjacent or opposite parts. For example, of the parts $\overline{A}$, b, and $\bar{c}$, the middle part is $\overline{A}$, and b and $\bar{c}$ adjacent parts; of the parts $\overline{B}$, $\bar{c}$, and b, the middle part is b and $\bar{c}$ and $\overline{B}$ are opposite parts.

If $\bar{c}$ is taken as a middle part; by the first rule

$$\sin \bar{c} = \cos c = \cos a \cos b, \tag{1}$$

by the second rule,

$$\sin \bar{c} = \cos c = \tan \overline{A} \cdot \tan \overline{B} = \cot A \cdot \cot B. \tag{2}$$

If a and b are taken as middle parts; by the first rule

$$\sin a = \cos \overline{A} \cdot \cos \bar{c} = \sin A \cdot \sin c, \tag{3}$$

$$\sin b = \cos \overline{B} \cdot \cos \bar{c} = \sin B \cdot \sin c; \tag{4}$$

by the second rule,

$$\sin a = \tan b \cdot \tan \overline{B} = \tan b \cdot \cot B, \tag{5}$$

$$\sin b = \tan a \cdot \tan A = \tan a \cdot \cot A. \tag{6}$$

If $\overline{A}$ and $\overline{B}$ are taken as middle parts; by the first rule

$$\sin \overline{A} = \cos A = \cos a \cdot \cos \overline{B} = \cos a \cdot \sin B, \tag{7}$$

$$\sin \overline{B} = \cos B = \cos b \cdot \cos \overline{A} = \cos b \cdot \sin A; \tag{8}$$

by the second rule,

$$\sin \overline{A} = \cos A = \tan b \tan \bar{c} = \tan b \cot c, \tag{9}$$

$$\sin \overline{B} = \cos B = \tan a \tan \bar{c} = \tan a \cot c. \tag{10}$$

The correctness of Napier's Rules can be established by showing that these ten formulae are correct.[1] This can be done by means of the Sine Law (48), the Cosine formulae (46) and (47), and by transformations among the results obtained from these two.

[1] For a more complete discussion of the derivation of Napier's Rules of Circular Parts, see Todhunter's *Spherical Trigonometry*, Arts. 66 to 70.

The equations (1), (2), (3), (4), (7), (8) can be obtained by putting $C=90°$ in the Sine Law and the Cosine formulae, wherever it occurs. For example, from (46)

$$\cos c=\frac{\cos A \cos B}{\sin A \sin B}=\cot A \cdot \cot B.$$

From (8), $\sin A=\frac{\cos B}{\cos b}$; from (4) $\sin c=\frac{\sin b}{\sin B}$; substituting these values in (3) we have,

$$\sin a=\frac{\cos B}{\cos b} \cdot \frac{\sin b}{\sin B}=\cot B \cdot \tan b. \tag{5}$$

Similarly, from (7), (3), and (4) we obtain (6). From (1) $\cos a=\frac{\cos c}{\cos b}$; from (4), $\sin B=\frac{\sin b}{\sin c}$; substituting in (7) we obtain,

$$\cos A=\frac{\cos c}{\cos b} \cdot \frac{\sin b}{\sin c}=\tan b \cot c. \tag{9}$$

Similarly, from (1), (3), and (8) we obtain (10). Thus the correctness of the rules is established.

110. Solution of the Right Triangle.—When two parts, other than the right angle, of a right spherical triangle are given the three unknown parts can be computed. As in other computation; (1) a rough triangle should be drawn and lettered with the given parts clearly marked, (2) the necessary formulae should be written down and numbered, (3) a complete form should be made for the computation, and (4) the necessary logarithms should be taken from the table. The following special suggestions are made.

All unknown parts should be computed directly from the given data and not from previously computed parts. Thus, to obtain formulae for computing, apply Napier's Rules to the two given parts together with each unknown part in turn.

A check formula can then be obtained by applying Napier's Rules to the three unknown parts. This check should be applied before looking up the angles since that operation is a waste of time if the logarithms are incorrect. The three final entries, that is, of the values of the unknown parts, are unchecked and should be made with extreme care.

Since in a spherical triangle an angle or side may have a value between 0° and 180°, a computed part will sometimes be found in the first quadrant and sometimes in the second. When a part is found from a cosine, tangent, or cotangent, the sign of the function determines the quadrant. When a computed part is found from a sine there will be a value in the first quadrant and one in the second.

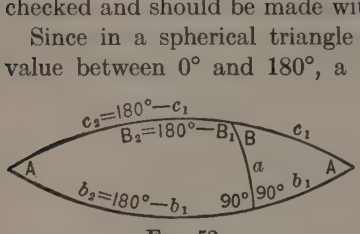


FIG. 52

In particular, when the two given parts are a leg and its opposite angle, there are always two solutions (unless the given data are absurd) and the given parts are found from sines. Figure 52 shows that if A and a are given we may construct either of the two triangles that form the lune.

111. Rules for Quadrants.—The theorems that follow will show how to select the values of the unknown parts in case a part is found from its sine.

Applying (49) to the parts B, A, and a of a right triangle, we take A as the middle part and B and a as the opposite parts. Then,

$$\sin \overline{A} = \cos \overline{B} \cdot \cos a \quad \text{or} \quad \cos A = \sin B \cdot \cos a;$$

since $B < 180°$, $\sin B$ is positive and, therefore, $\cos A$ and $\cos a$ have the same sign and a and A are in the same quadrant. Thus: **In a right spherical triangle an oblique angle and the side opposite are in the same quadrant.**

From the equation $\cos c = \cos a \cdot \cos b$ it appears that $\cos c$ is positive when $\cos a$ and $\cos b$ agree in sign and is negative

when they have unlike signs. Since cos a and cos b can agree in sign only when a and b are in the same quadrant we conclude that **the hypotenuse of a right spherical triangle is less than 90° when the two legs are in the same quadrant and is greater than 90° when the two legs are in opposite quadrants and conversely.**

112. Examples.

1. In a right spherical triangle, given $a = 116° 24' 25''$, $b = 16° 50' 30''$, find the remaining parts.

First we write the necessary formulae with the aid of (49).

For c, $\sin \bar{c} = \cos a \cos b$, or $\cos c = \cos a \cos b$; (1)
for A, $\sin b = \tan a \tan \bar{A}$, or $\cot A = \sin b \cot a$; (2)
for B, $\sin a = \tan b \tan \bar{B}$, or $\cot B = \sin a \cot b$; (3)
check, $\sin \bar{c} = \tan \bar{A} \tan \bar{B}$, or $\cos c \tan A \tan B = 1$.

	°	′	″	(1)	(2)	(3)	*Check*
a,	116	24	25	cos(−) 9.64811	cot(−) 9.69597	sin 9.95214	
b,	16	50	30	cos 9.98096	sin 9.46199	cot 0.51897	
c,	115	11	34	cos(−) 9.62907			cos(−) 9.62907
A,	98	11	13		cot(−) 9.15796		tan(−) 0.84204
B,	18	40	27			cot 0.47111	tan 9.52889
							0.00000

2. In a right spherical triangle, given $A = 35° 20' 00''$, $c = 103° 10' 00''$; find the remaining parts.

From (49) we have,

for B, $\sin \bar{c} = \tan \bar{B} \tan \bar{A}$ or $\cot B = \cos c \tan A$; (1)
for b, $\sin \bar{A} = \tan b \tan \bar{c}$ or $\tan b = \cos A \tan c$; (2)
for a, $\sin a = \cos \bar{c} \cos \bar{A}$ or $\sin a = \sin c \sin A$; (3)
check, $\sin a = \tan b \cot B$.

If the middle part of the check formula is computed in the third column, the check can be put in the first column as is shown in the following:

	°	′	″	(1)	(2)	(3)
A,	35	20	00	tan 9.85059	cos 9.91158	sin 9.76218
c,	103	10	00	cos(−) 9.35752	tan(−) 0.63091	sin 9.98843
B,	99	10	22	cot(−) 9.20811		
b,	106	00	01	tan(−) 0.54249	tan(−) 0.54249	
a,	34	16	23	sin 9.75060	← *check* →	sin 9.75061

In this solution a is found from its sine and might be in the second quadrant but for the fact that $A<90°$.

3. In a right spherical triangle, given $B=126°\ 57'\ 37''$, $b=154°\ 00'\ 32''$; find the remaining parts.

For A, $\sin \overline{B}=\cos b \cos \overline{A}$ or $\sin A=\cos B \sec b$; (1)
for c, $\sin b=\cos \overline{c} \cos \overline{B}$ or $\sin c=\sin b \csc B$; (2)
for a, $\sin a=\tan b \tan \overline{B}$, or $\sin a=\tan b \cot B$; (3)
check, $\sin a=\cos \overline{A} \cos \overline{c}$ or $\sin a=\sin A \sin c$.

	(1)	(2)	(3)	*Second triangle*
B, 126 57 37	cos(−) 9.77907	csc 0.09743	cot(−) 9.87649	
b, 154 00 32	sec(−) 0.04631	sin 9.64170	tan(−) 9.68801	
A_1, 41 59 05	sin 9.82538			A_2, 138 00 55
c_1, 146 44 22	sin 9.73913	sin 9.73913		c_2, 33 15 38
a_1, 21 31 19	sin 9.56451	← *check* →	sin 9.56450	a_2, 158 28 41

One value of a is marked a_1 and the other a_2; c_1 is to be associated with a_1 and, therefore, must be obtuse, since a_1 and b are in different quadrants; since b and a_2 are both obtuse, c_2 must be acute; also A_1 must be in the same quadrant with a_1 and A_2 with a_2.

EXERCISE XLIX

Solve the following right spherical triangles, given

1. $a=60°\ 00'\ 00''$, $c=69°\ 17'\ 42''$.
2. $a=51°\ 12'\ 00''$, $b=120°\ 25'\ 32''$.
3. $c=103°\ 10'\ 00''$, $B=99°\ 10'\ 22''$.
4. $a=141°\ 03'\ 54''$, $b=74°\ 19'\ 37''$.
5. $A=55°\ 18'\ 13''$, $B=69°\ 55'\ 20''$.
6. $A=69°\ 30'\ 45''$, $b=119°\ 30'\ 40''$.
7. $b=59°\ 18'\ 27''$, $c=106°\ 18'\ 32''$.
8. $a=67°\ 06'\ 53''$, $A=80°\ 10'\ 30''$.
9. $A=60°\ 47'\ 24''$, $B=57°\ 16'\ 20''$.
10. $a=134°\ 28'\ 13''$, $A=125°\ 22'\ 40''$.
11. $A=147°\ 46'\ 28''$, $a=159°\ 17'\ 55''$.

12. $A = 119° 08' 42''$, $a = 121° 27' 23''$.

13. $c = 109° 41' 50''$, $B = 27° 36' 59''$.

14. $B = 80° 00' 00''$, $b = 68° 00' 00''$.

15. $a = 67° 06' 53''$, $b = 153° 58' 25''$.

16. $B = 152° 23' 01''$, $b = 154° 07' 27''$.

17. $A = 17° 41' 40''$, $B = 97° 39' 24''$.

18. $a = 51° 43' 36''$, $b = 54° 32' 32''$.

19. Solve the isosceles spherical triangle, given $A = B = 55° 28' 30''$, $C = 78° 29' 41''$. (Drop a perpendicular and divide the triangle into two right triangles.)

20. Solve the isosceles spherical triangle, given $a = b = 129° 26' 37''$, $A = 99° 45' 22''$.

113. The Quadrantal Triangle.—A spherical triangle with one side equal to 90° is called a quadrantal triangle. Its polar triangle is a right triangle since the angle opposite the quadrantal side is 90°. Thus the solution of a quadrantal triangle can readily be accomplished by solving its polar triangle. The following example indicates the method.

Example.—Solve the spherical triangle in which $c = 90°$, $b = 104° 30' 00''$, $C = 61° 39' 00''$.

In the polar triangle
$C' = 180° - c = 90°$, $B' = 180° - b = 75° 30' 00''$,
$c' = 180° - C = 118° 21' 00''$.
Then
$\sin \overline{c}' = \tan \overline{A}' \tan \overline{B}'$, or $\cot A' = \cos c' \tan B'$; (1)
$\sin \overline{B}' = \tan a' \tan \overline{c}'$, or $\tan a' = \tan c' \cos B'$; (2)
$\sin b' = \cos \overline{c}' \cos \overline{B}'$, or $\sin b' = \sin c' \sin B'$; (3)
Check,
$\sin b' = \tan a' \tan A'$, or $\sin b' = \tan a' \cot A'$.

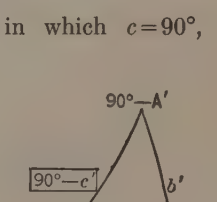


FIG. 53

		(1)	(2)	(3)	*Original triangle*
B',	75 30 00	tan 0.58734	cos 9.39860	sin 9.98594	° ′ ″
c',	118 21 00	cos(−) 9.67656	tan(−) 0.26795	sin 9.94451	
A',	151 25 34	cot(−) 0.26390			a = 28 34 26
a',	155 06 25	tan(−) 9.66655	tan(−) 9.66655		A = 24 53 35
b',	58 25 54	sin 9.93045	← *check* →	sin 9.93045	B =121 34 06

EXERCISE L

Solve the following spherical triangles; given

1. $C=90°$, $a=90°$, $B=60°$.
2. $C=90°$, $A=90°$, $B=103°$.
3. $c=90°$, $a=83°\ 36'\ 12''$, $A=52°\ 14'\ 15''$.
4. $c=90°$, $a=99°\ 49'\ 20''$, $B=24°\ 14'\ 15''$.
5. $c=90°$, $A=63°\ 35'\ 35''$, $B=163°\ 09'\ 30''$.
6. $c=90°$, $C=44°\ 19'\ 40''$, $A=159°\ 39'\ 40''$.
7. $c=90°$, $a=53°\ 39'\ 00''$, $A=42°\ 30'\ 00''$.
8. $c=90°$, $A=85°\ 20'\ 00''$, $C=93°\ 30'\ 00''$.

CHAPTER XIII

SOLUTION OF THE OBLIQUE SPHERICAL TRIANGLE

Before discussing the solution of the oblique spherical triangle we shall derive other useful formulae from the Sine Law and Cosine Law.

114. The Half-Angle Formulae.—By (45)

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \cdot \sin c};$$

subtracting both sides from 1,

$$1 - \cos A = \frac{\sin b \cdot \sin c + \cos b \cdot \cos c - \cos a}{\sin b \cdot \sin c},$$

$$= \frac{\cos (b-c) - \cos a}{\sin b \sin c}, \qquad \text{(By 13)}$$

$$= \frac{2 \sin \frac{a+b-c}{2} \cdot \sin \frac{a-b+c}{2}}{\sin b \sin c} \qquad \text{(By 30);}$$

if $2s = a+b+c$, then $\frac{a+b-c}{2} = s-c$, $\frac{a-b+c}{2} = s-b$,

and we may write $1 - \cos A = \frac{2 \sin (s-b) \cdot \sin (s-c)}{\sin b \cdot \sin c}$,

or

$$\text{hav A} = \sin^2 \frac{A}{2} = \frac{\sin (s-b) \cdot \sin (s-c)}{\sin b \cdot \sin c} \quad \text{(By 20).} \qquad (50)$$

Similarly, $1 + \cos A = \frac{\sin b \cdot \sin c - \cos b \cdot \cos c + \cos a}{\sin b \cdot \sin c}$,

$$= \frac{\cos a - \cos (b+c)}{\sin b \sin c} \qquad \text{(By 10),}$$

$$1+\cos A=\frac{2\sin\frac{a+b+c}{2}\cdot\sin\frac{b+c-a}{2}}{\sin b\cdot\sin c}\qquad\text{(By 30)},$$

$$=\frac{2\sin s\cdot\sin(s-a)}{\sin b\cdot\sin c};$$

and
$$\cos^2\frac{A}{2}=\frac{\sin s\cdot\sin(s-a)}{\sin b\cdot\sin c}\qquad\text{(By 21).}\qquad(51)$$

Dividing (50) by (51), we obtain

$$\tan^2\frac{A}{2}=\frac{\sin(s-b)\cdot\sin(s-c)}{\sin s\cdot\sin(s-a)};\qquad(52)$$

multiplying numerator and denominator by $\sin(s-a)$, and taking the square root, we have

$$\tan\frac{A}{2}=\frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\cdot\sin(s-b)\cdot\sin(s-c)}{\sin s}}$$

$$=\frac{P}{\sin(s-a)},\qquad(53)$$

where
$$P=\sqrt{\frac{\sin(s-a)\cdot\sin(s-b)\cdot\sin(s-c)}{\sin s}}.\qquad(54)$$

Similarly, $\tan\frac{B}{2}=\frac{P}{\sin(s-b)}$, $\tan\frac{C}{2}=\frac{P}{\sin(s-c)}$.

115. Napier's Analogies.—By (9)

$$\sin\tfrac{1}{2}(A+B)=\sin\frac{A}{2}\cdot\cos\frac{B}{2}+\cos\frac{A}{2}\cdot\sin\frac{B}{2},$$

$$=\sqrt{\frac{\sin(s-b)\cdot\sin(s-c)}{\sin b\cdot\sin c}}\sqrt{\frac{\sin s\cdot\sin(s-b)}{\sin a\cdot\sin c}}$$

$$+\sqrt{\frac{\sin s\cdot\sin(s-a)}{\sin b\cdot\sin c}}\sqrt{\frac{\sin(s-a)\cdot\sin(s-c)}{\sin a\cdot\sin c}}\qquad\text{(By 50, 51),}$$

$$= \frac{\sin(s-b)+\sin(s-a)}{\sin c}\sqrt{\frac{\sin s \cdot \sin(s-c)}{\sin a \cdot \sin b}};$$

likewise

$$\sin \tfrac{1}{2}(A-B) = \frac{\sin(s-b)-\sin(s-a)}{\sin c}\sqrt{\frac{\sin s \cdot \sin(s-c)}{\sin a \cdot \sin b}};$$

dividing

$$\frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} = \frac{\sin(s-b)-\sin(s-a)}{\sin(s-b)+\sin(s-a)},$$

$$= \frac{2\cos \frac{c}{2} \cdot \sin \frac{1}{2}(a-b)}{2\sin \frac{c}{2} \cdot \cos \frac{1}{2}(a-b)} \qquad \text{(By 27, 28)}$$

or

$$\frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} = \frac{\tan \frac{1}{2}(a-b)}{\tan \frac{c}{2}}. \qquad (55)$$

Similarly,

$$\cos \tfrac{1}{2}(A \pm B) = \cos \frac{A}{2} \cdot \cos \frac{B}{2} \mp \sin \frac{A}{2} \cdot \sin \frac{B}{2} \qquad \text{(By 10, 13),}$$

$$= \sqrt{\frac{\sin s \cdot \sin(s-a)}{\sin b \cdot \sin c}}\sqrt{\frac{\sin s \cdot \sin(s-b)}{\sin a \cdot \sin c}}$$

$$\mp \sqrt{\frac{\sin(s-b) \cdot \sin(s-c)}{\sin b \cdot \sin c}}\sqrt{\frac{\sin(s-a) \cdot \sin(s-c)}{\sin a \cdot \sin c}}, \qquad \text{(By 50, 51),}$$

$$= \frac{\sin s \mp \sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a) \cdot \sin(s-b)}{\sin a \cdot \sin b}};$$

and by division

$$\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} = \frac{\sin s+\sin(s-c)}{\sin s-\sin(s-c)},$$

$$= \frac{2\sin \frac{1}{2}(a+b)\cos \frac{c}{2}}{2\cos \frac{1}{2}(a+b)\sin \frac{c}{2}} \qquad \text{(By 27);}$$

or

$$\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{c}{2}}. \qquad (56)$$

If ABC and $A'B'C'$ are two polar triangles,

$\cos \frac{1}{2}(A'+B') = \cos [180° - \frac{1}{2}(a+b)] = -\cos \frac{1}{2}(a+b)$,

$\sin \frac{1}{2}(A'+B') = \sin [180° - \frac{1}{2}(a+b)] = \sin \frac{1}{2}(a+b)$,

$\cos \frac{1}{2}(A'-B') = \cos \frac{1}{2}(a-b)$,

$\sin \frac{1}{2}(A'-B') = \sin \frac{1}{2}(b-a) = -\sin \frac{1}{2}(a-b)$,

$\tan \frac{1}{2}(a'+b') = \tan [180° - \frac{1}{2}(A+B)] = -\tan \frac{1}{2}(A+B)$,

$\tan \frac{1}{2}(a'-b') = \tan \frac{1}{2}(B-A) = -\tan \frac{1}{2}(A-B)$,

$\tan \frac{c'}{2} = \tan (90° - C/2) = \cot C/2$; from (55) and (56)

$$\frac{\sin \frac{1}{2}(A'-B')}{\sin \frac{1}{2}(A'+B')} = \frac{\tan \frac{1}{2}(a'-b')}{\tan \frac{c'}{2}}, \quad \frac{\cos \frac{1}{2}(A'-B')}{\cos \frac{1}{2}(A'+B')} = \frac{\tan \frac{1}{2}(a'+b')}{\tan \frac{c'}{2}}$$

and substituting for the primes, we have

$$\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2}C}, \tag{57}$$

$$\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A+B)}{\cot \frac{1}{2}C}. \tag{58}$$

Equations (55) to (58) are known as Napier's Analogies.

116. The student must have noticed the analogy between formulae of Spherical trigonometry and formulae of Plane trigonometry. It is possible to deduce formulae for solving plane triangles from the formulae for spherical triangles.

Let $a = Ra$, $\beta = Rb$, $\gamma = Rc$ be the linear lengths (see Art. 21) of the sides of a triangle ABC on a sphere of radius R. Then $a = a/R$, $b = \beta/R$, $c = \gamma/R$, and we may write the Sine Law,

$$\frac{\sin A}{\sin a/R} = \frac{\sin B}{\sin \beta/R} = \frac{\sin C}{\sin \gamma/R}.$$

If R increases indefinitely the spherical surface tends to become a plane and $\sin \frac{a}{R} \doteq \frac{a}{R}$ (see Art. 55).

Thus we may substitute the radian measure of the side for the sine of the side and obtain

$$\frac{\sin A}{\alpha/R}=\frac{\sin B}{\beta/R}=\frac{\sin C}{\gamma/R},$$

or

$$\frac{\sin A}{\alpha}=\frac{\sin B}{\beta}=\frac{\sin C}{\gamma},$$

which is the Sine Law for plane triangles.

In a similar way we can write formula (57) as

$$\frac{\tan \tfrac{1}{2}(A-B)}{\cot C/2}=\frac{\sin \tfrac{1}{2}(\alpha/R-\beta/R)}{\sin \tfrac{1}{2}(\alpha/R+\beta/R)},$$

as $R \doteq \infty$, this becomes

$$\frac{\tan \tfrac{1}{2}(A-B)}{\cot C/2}=\frac{\tfrac{1}{2}(\alpha/R-\beta/R)}{\tfrac{1}{2}(\alpha/R+\beta/R)}=\frac{\alpha-\beta}{\alpha+\beta},$$

which is the Tangent Law for plane triangles. In a similar manner other formulae for plane triangles may be deduced.

EXERCISE LI

1. By using the properties of polar triangles derive from (50) and (51) the relations,

$$\text{hav } a=\sin^2\frac{a}{2}=\frac{-\cos S\cdot\cos (S-A)}{\sin B\cdot\sin C}$$

and

$$\cos^2\frac{a}{2}=\frac{\cos (S-B)\cdot\cos (S-C)}{\sin B\sin C},$$

where $S=\tfrac{1}{2}(A+B+C)$.

2. Show that $\tan\dfrac{a}{2}=P'\cdot\cos (S-A)$ where

$$P'=\sqrt{\frac{-\cos S}{\cos (S-A)\cdot\cos (S-B)\cdot\cos (S-C)}}.$$

3. Show that P, formula (54), is equal to $\tan r$ where r is the radius of the small circle inscribed in the triangle ABC.

4. Show that the quantity P' of Ex. 2 is $\tan R$ where R is

the radius of the small circle circumscribed about the triangle ABC.

5. Show that $\dfrac{\sin a}{\sin A} = -\sin s \cdot \tan r \cdot \tan R \cdot \sec S$.

6. Show that $\text{hav } a = \dfrac{\sin E/2 \cdot \sin (A - E/2)}{\sin B \cdot \sin C}$ and

$$\cos \frac{a}{2} = \sqrt{\frac{\sin (B - E/2) \cdot \sin (C - E/2}{\sin B \cdot \sin C}}.$$

7. Show that

$$\tan \frac{E}{4} = \sqrt{\tan \tfrac{1}{2}s \cdot \tan \tfrac{1}{2}(s-a) \cdot \tan \tfrac{1}{2}(s-b) \cdot \tan \tfrac{1}{2}(s-c)}.$$

8. Show that

$$\text{hav } E = \frac{\sin s \cdot \sin (s-a) \cdot \sin (s-b) \cdot \sin (s-c)}{4 \text{ hav } (180° - a) \cdot \text{hav } (180° - b) \cdot \text{hav } (180° - c)}.$$

9. Prove the Sine Law by writing $\dfrac{\sin A}{\sin a} = \dfrac{2 \sin \dfrac{A}{2} \cdot \cos \dfrac{A}{2}}{\sin a}$

and substituting from (50) and (51).

10. From Art. 115 obtain Gauss's equations

$$\frac{\sin \frac{1}{2}(A+B)}{\cos C/2} = \frac{\cos \frac{1}{2}(a-b)}{\cos c/2},$$

$$\frac{\cos \frac{1}{2}(A+B)}{\sin C/2} = \frac{\cos \frac{1}{2}(a+b)}{\cos c/2},$$

$$\frac{\sin \frac{1}{2}(A-B)}{\cos C/2} = \frac{\sin \frac{1}{2}(a-b)}{\sin c/2},$$

$$\frac{\cos \frac{1}{2}(A-B)}{\sin C/2} = \frac{\sin \frac{1}{2}(a+b)}{\sin c/2}.$$

11. Show that the third of Gauss's equations can be written

$$\text{hav } (A-B) = \frac{\text{hav } (a-b)}{\text{hav } c} \cdot \text{hav } (180° - C).$$

12. If $a=134° 28' 14''$, $b=128° 32' 14''$, $c=49° 18' 29''$; find E and A.

13. If $c=46° 30' 29''$, $b=30° 20' 17''$, $A=35° 59' 28''$; find C and B by Napier's Analogies.

14. If $A=61° 37' 53''$, $B=139° 54' 34''$, $C=26° 21' 18''$; find a and c.

15. Given $a=99° 40' 48''$, $b=100° 49' 30''$, $c=65° 33' 10''$; find A, B, C.

16. Given $A=80° 19' 12''$, $B=79° 10' 30''$, $c=114° 26' 50''$; find a and b.

17. Show from Napier's Analogies that in any spherical triangle one-half the sum of two angles is in the same quadrant as one-half the sum of the sides opposite; that is, $\frac{a+b}{2}$ and $\frac{A+B}{2}$ are in the same quadrant.

In a right spherical triangle, prove the following relations:

18. $\tan^2\left(45°-\frac{B}{2}\right)=\cot \frac{1}{2}(b+c)\cdot\tan \frac{1}{2}(c-b)$.

19. $\text{hav}\,(90°+B)=\sin \frac{1}{2}(b+c)\cdot\cos \frac{1}{2}(c-b)\cdot\csc c$.

20. $\tan^2\frac{a}{2}=\tan \frac{1}{2}(c+b)\cdot\tan \frac{1}{2}(c-b)$.

21. $\text{hav}\,a=\sin \frac{1}{2}(b+c)\cdot\sin \frac{1}{2}(c-b)\cdot\sec b$.

22. $\tan^2\frac{A}{2}=\frac{\sin \frac{1}{2}(c-b)}{\sin \frac{1}{2}(c+b)}$.

23. $\text{hav}\,A=\frac{1}{2}\sin (c-b)\cdot\sec b\cdot\csc c$.

Solve the following triangles

24. $c = 90°$, $B = 118°\ 26'\ 34''$, $C = 117°\ 56'\ 19''$ (use formulae from Examples 18–23).

25. $C = 90°$, $a = 85°\ 20'\ 19''$, $c = 86°\ 30'\ 19''$ (use formulae from Examples 18–23).

117. A spherical triangle is determined when any three of its six parts are known. Thus in the solution of the oblique spherical triangle the following six cases arise:

I. Given two sides and the included angle.

II. Given two angles and the included side.

III. Given two sides and an angle opposite one of them.

IV. Given two angles and a side opposite one of them.

V. Given three sides.

VI. Given three angles.

The Sine Law (48), Napier's Analogies (55)–(58), and the formulae for the tangent of a half-angle (53) and (54), or the haversine of an angle (50), comprise sufficient formulae for the solution of all these cases. It should be noted, too, that by use of the polar triangle, Cases II, IV, and VI can be reduced to Cases I, III, and V, respectively. However, only in Case VI is there a gain in the use of the polar triangle.

118. It is also possible to make the solution of the oblique triangle depend upon the solution of the right triangle by drawing a perpendicular from a vertex to the opposite side and solving by Napier's Rules the two right triangles so formed. Certain general directions follow for carrying out this method.

The perpendicular should be drawn, if possible, so that two given parts come in one (the first) of the right triangles formed and two required parts in the other (the second). In Cases I and II the perpendicular can be drawn in two ways, in Cases III and IV in only one way. The perpendicular method is not of particular value in Cases V and VI.

To aid in writing formulae draw a plane triangle, denoting the perpendicular by p, the segments into which it divides the

base by ϕ and ϕ' and the segments of the divided angle by θ and θ' as shown in Fig. 54. The primed letters are in the second triangle.

The quadrant of a computed part is determined as in the solution of the right triangle, hence ϕ and θ are always in the same quadrant; the same is true of ϕ' and θ'.

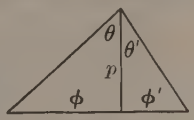


FIG. 54

In deriving the formulae for working any problem by this method it is sufficient to draw a simple triangle as shown in Fig. 54. Of course if the perpendicular falls outside, and the two right triangles overlap, as in Fig. 55, ϕ or ϕ' must be taken as negative in order that $\phi+\phi'$ shall give the length of the base. Either of the angles ϕ or ϕ' may be taken as a positive

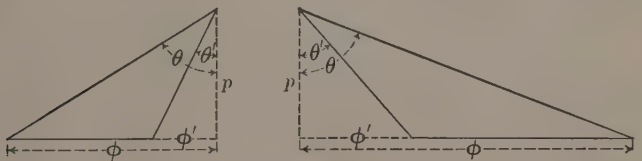


FIG. 55

angle in the first or second quadrant, or as a negative angle in the third or fourth quadrant, depending on the sign of the function by which the angle is determined. Therefore ϕ and ϕ', and consequently, θ and θ', are never taken numerically greater than 180°. However, no difficulty will be found when ϕ or ϕ' is in the third or fourth quadrant, if θ be always taken in the same quadrant with ϕ and θ' in the same quadrant with ϕ'.

It is not necessary to find the value of p since it can be eliminated from the required formulae.

The Sine Law is reserved for a check in all cases.

119. Case I: Given a, b, and C.—Three methods for solving this case will be given: (1) by dropping a perpendicular, (2) by

the haversine formula (50), (3) by Napier's Analogies. The method (1) is especially advantageous for finding the two parts, c and B, or c and A, (2) is the best method when c alone is required, and (3) is the shortest method for finding all the unknown parts of the triangle.

There are two ways in which the perpendicular can be drawn in case I, but if only one unknown angle is required, the perpendicular should be drawn from the vertex of the angle not required.

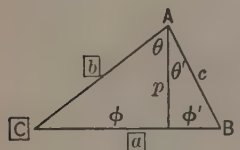


FIG. 56

Thus, if c and B are required drop the perpendicular from A as in Fig. 56. From the first right triangle,

$$\sin \bar{C} = \tan \phi \tan \bar{b},$$
$$\tan \phi = \cos C \tan b. \quad (1)$$
Then $\phi' = a - \phi.$ (2)

To find c, select b, ϕ, p in the first triangle and c, ϕ', p in the second and apply Napier's Rules, obtaining

$\sin \bar{b} = \cos b = \cos p \cdot \cos \phi$ and $\sin \bar{c} = \cos c = \cos p \cdot \cos \phi'$; eliminating p,

$$\cos c = \cos b \cdot \sec \phi \cdot \cos \phi'. \quad (3)$$

To find B; select C, ϕ, p in the first triangle and B, ϕ', p in the second, and obtain,

$$\sin \phi = \cot C \cdot \tan p, \ \sin \phi' = \cot B \cdot \tan p;$$

eliminating p,

$$\cot B = \sin \phi' \csc \phi \cot C. \quad (4)$$

If A is not required we write a check formula connecting c, B and ϕ'

$$\sin \bar{B} = \cos B = \tan \phi' \cot c. \quad (5)$$

To find A; from the first triangle

$$\sin \bar{b} = \tan \bar{\theta} \tan \bar{C} \text{ or } \cot \theta = \cos b \tan C, \quad (6)$$

from the second triangle

$$\sin \bar{c} = \tan \bar{\theta}' \tan \bar{B} \text{ or } \cot \theta' = \cos c \cdot \tan B; \quad (7)$$

and
$$A = \theta + \theta'. \quad (8)$$

NOTE: In every case where a formula is derived from parts in one triangle and associated with a similar formula derived from parts in the other triangle, thus eliminating p, symmetrical formulae are used. Thus to get (3), we associate b, p, ϕ in one and c, p, ϕ' in the other triangle; to get (4) we associate ϕ, C, p in one and ϕ', B, p in the other, etc. In this way the function of p is always the same in each formula.

The complete solution is checked by the Sine formula.

If $\tan \phi$ is negative we shall take ϕ in the second quadrant and ϕ' will be in the fourth quadrant. However, we may take ϕ in the fourth quadrant and then ϕ' will be in the second. In either case θ and θ' are in the same quadrants as ϕ and ϕ' respectively.

In solving a spherical triangle the importance of making a complete form for the computation, before taking logarithms from the tables, is greater than in any previous computation. In Art. 119 (a) a form for the computation is given.

120. Case I: Haversine Solution.—When a, b, and C are given, the third side c can readily be found by the formula (47).

$$\text{hav } c = \text{hav } (b-a) + \sin b \cdot \sin a \cdot \text{hav } C.$$

The angles A and B can be found by the Sine Law.

The computation is shown below for the Example of 119 (a).

	°	′	″					
C,	61	18	27		hav 9.41488	sin 9.94310	sin 9.94310	
b,	132	14	23		sin 9.86943	sin 9.86943		
a,	63	29	56		sin 9.95179		sin 9.95179	
				N. hav 0.17223	hav 9.23610			
$b-a$,	68	44	27	N. hav 0.31871				
c,	88	57	41	N. hav 0.49094		csc 0.00007	csc 0.00007	
						sin 9.81260		
B,	139	29	36					
A,	51	44	07				sin 9.89496	

Table 45 can be used as a table of logarithms of numbers as is shown in this example for log. $0.17223 = 9.23610$.

Since $a < c < b$ we know that A and B must be in such quadrants that $A < C < B$.

119(a). Example: Solve the spherical triangle, given $b = 132° 14' 23''$, $a = 63° 29' 56''$, $C = 61° 18' 27''$.

$$\tan \phi = \cos C \cdot \tan b \quad (1)$$
$$\cos c = \cos b \sec \phi \cos \phi' \quad (2)$$
$$\cot B = \sin \phi' \csc \phi \cot C \quad (3)$$
$$\cot \vartheta = \cos b \cdot \tan C \quad (4)$$
$$\cot \theta' = \cos c \cdot \tan B \quad (5)$$
$$\cos B = \tan \phi' \cot c \quad (6)$$
$$\phi' = a - \phi \quad (7)$$
$$A = \theta + \theta' \quad (8)$$

	° ′ ″	(1) & (6)	(2)	(3)	(4)	(5)
$C =$	61 18 27	cos		cot	tan	
$b =$	132 14 23	tan(−)	cos(−)		cos(−)	
$a =$	63 29 56					
$\phi =$	152 07 56	tan(−) 9.72325	sec(−)	csc		
$\phi' =$	−(88 38 00)	tan(−)	cos	sin(−)		
$c =$	88 57 39	cot	cos 8.25855			cos
$B =$	139 29 33	cos(−)		cot(−)0.06839		tan(−)
$\vartheta =$	140 50 56				cot(−)0.08929	
$\theta' =$	−(89 06 44)					cot(−)8.19016
$A =$	51 44 12					

	$\frac{\sin A}{\sin a}$	$\frac{\sin B}{\sin b}$	$\frac{\sin C}{\sin c}$
Check	9.94318	9.94318	9.94318

121. Case I: By Napier's Analogies.—Case I can be solved very readily by Napier's Analogies (55)–(58). The solution of the example given in Art. 119 (a) is shown below.

$$\tan \tfrac{1}{2}(B+A) = \frac{\cos \tfrac{1}{2}(b-a)}{\cos \tfrac{1}{2}(b+a)} \cot C/2. \tag{1}$$

$$\tan \tfrac{1}{2}(B-A) = \frac{\sin \tfrac{1}{2}(b-a)}{\sin \tfrac{1}{2}(b+a)} \cot C/2. \tag{2}$$

$$\tan c/2 = \frac{\cos \tfrac{1}{2}(B+A)}{\cos \tfrac{1}{2}(B-A)} \tan \tfrac{1}{2}(b+a). \tag{3}$$

$$\tan c/2 = \frac{\sin \tfrac{1}{2}(B+A)}{\sin \tfrac{1}{2}(B-A)} \tan \tfrac{1}{2}(b-a). \qquad \text{(Check)}$$

	° ′ ″	(1)	(2)	(3)	Check
C	61 18 27				
$\frac{C}{2}$	30 39 14	cot 0.22719	cot 0.22719		
b	132 14 23				
a	63 29 56				
$\frac{1}{2}(b-a)$	34 22 14	cos 9.91667	sin 9.75169		tan 9.83503
$\frac{1}{2}(b+a)$	97 52 10	sec(−) 0.86354	csc 0.00411	tan(−) 0.85943	
$\frac{1}{2}(B+A)$	95 36 53	tan(−)11.00740		cos(−) 8.99053	sin 9.99791
$\frac{1}{2}(B-A)$	43 52 42		tan 9.98299	sec 0.14217	csc 0.15919
$\frac{c}{2}$	44 28 52			tan 9.99213	tan 9.99213
c	88 57 44				
B	139 29 35				
A	51 44 11				

EXERCISE LII

1. In a spherical triangle, given $b = 118° 42' 00''$, $A = 65° 29' 00''$, $c = 68° 34' 25''$; solve the triangle completely by dropping a perpendicular through C and check by the Sine formula.

2. Solve the triangle of Example **1** by Napier's Analogies.

3. In a spherical triangle, given $b = 75° 18' 00''$, $A = 68° 22' 00''$, $c = 63° 35' 41''$; find B, a, C.

4. In a spherical triangle given $b = 59° 19' 00''$, $A = 63° 28' 00''$, $c = 121° 48' 00''$; find B, a and C.

5. In a spherical triangle, given $b = 24° 42' 30''$, $A = 124° 24' 50''$, $c = 98° 03' 22''$; find B, a, C.

6. In a spherical triangle, given $b = 145° 28' 42''$, $A = 40° 19' 03''$, $c = 116° 07' 55''$; find B, a, C.

7. In a spherical triangle, given $A = 65° 33' 10''$, $b = 99° 40' 48''$, $c = 100° 49' 30''$; find a by formula (47).

8. In a spherical triangle, given $A = 117° 12' 20''$, $b = 52° 18' 15''$, $c = 68° 20' 25''$; find a by (47) and B and C by (48).

9. If a spherical triangle is to be constructed from the given parts a, b and C, are there any restrictions as to the values which may be assigned them?

122. Case II: Given A, B, c.—This case can be solved in three ways: (1) by dividing the triangle into two right triangles by a perpendicular through B or A, (2) by Napier's Analogies (55)–(58), or (3) by solving its polar triangle as Case I.

The formulae for the solution by the first method follow:—

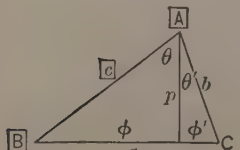


FIG. 57

$$\cot\theta = \cos c \cdot \tan B, \tag{1}$$

$$\theta' = A - \theta, \tag{2}$$

$$\cot b = \cot c \sec\theta \cos\theta', \tag{3}$$

$$\cos C = \cos B \csc\theta \sin\theta', \tag{4}$$

$$(check)\ \cos b = \cot C \cot\theta', \tag{5}$$

$$\tan\phi = \cos B \cdot \tan c, \tag{6}$$

$$\tan\phi' = \cos C \cdot \tan b, \tag{7}$$

$$a = \phi + \phi' \tag{8}$$

Check by Sine formula.

EXERCISE LIII

1. Derive the formulae given in Art. 122 from Fig. 57.

2. If only the third side and one unknown angle are desired, how should the perpendicular be drawn?

3. In a spherical triangle, given $A = 31° 34' 26''$, $B = 30° 28' 12''$, $c = 70° 02' 03''$, solve the triangle completely by the formulae of Art. 122 and check by the Sine formula.

4. Solve the triangle of Ex. 3 by Napier's Analogies.

5. Solve the triangle of Ex. 3 by solving its polar triangle by the formulae of Art. 119.

6. In a spherical triangle, given $B = 71° 55' 45''$, $a = 64° 21' 15''$, $C = 111° 06' 15''$, drop a perpendicular and find the values of A and b.

7. Given the triangle of Ex. 6. Compute A by the formula $\text{hav}\,(180°-A)=\text{hav}\,(B+C)-\sin B\cdot\sin C\cdot\text{hav}\,a$.

8. In a spherical triangle given $C=145°\ 24'\ 40''$, $A=111°\ 45'\ 50''$, $b=71°\ 37'\ 20''$, solve by drawing a perpendicular.

9. Solve the triangle of Ex. 8 by drawing a perpendicular through C' of its polar triangle.

10. In a spherical triangle given $A=91°\ 36'\ 00''$, $B=123°\ 12'\ 00''$, $c=51°44'\ 00''$, find the other parts.

11. In a spherical triangle, given $B=134°\ 51'\ 52''$, $c=76°\ 51'\ 23''$, $A=112°\ 19'\ 31''$; find the remaining parts.

12. If a spherical triangle is to be constructed from the given parts A, B, c, are there any restrictions on the values which may be assigned them?

123. Case III: Given a, b, A.—Case III is solved by dividing the triangle into two right triangles by a perpendicular and applying Napier's Rules.

As two known parts are to appear in the first triangle the perpendicular must be drawn from C. The formulae for the solution are:

$\tan\phi=\cos A\tan b$, (1)

$\cos\phi'=\cos\phi\cos a\sec b$, (2)

$c=\phi+\phi'$ where $\phi'=\phi_1'$ and ϕ_2', (3)

$\cot B=\cot A\cdot\sin\phi'\csc\phi$, (4)

$\cot\theta=\cos b\cdot\tan A$, (5)

$\cot\theta'=\cos a\cdot\tan B$, (6)

$C=\theta+\theta'$ where $\theta'=\theta_1'$, and θ_2' (7)

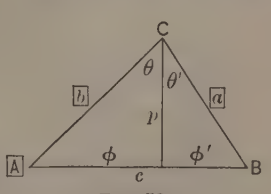


FIG. 58

Check by Sine formula.

In this case there may be two solutions. The auxiliary ϕ' is found from its cosine and may be positive or negative. In our formulae we denote these two values by ϕ' and two values of B, θ' and c are found by using them. However, $0°<c<180°$ and any values of ϕ' for which $\phi+\phi'>180°$ or $\phi+\phi'<0°$, consequently, must be discarded. This determines the proper number of solutions.

The computation with check by the Sine formula is shown in Art. 123 (a).

123(a). **Example**: In a spherical triangle given $a = 109° 41' 40''$, $b = 95° 38' 40''$, $A = 132° 36' 30''$, find the other parts.

$$\tan\phi = \cos A \tan b \quad (1)$$
$$\cos\phi' = \cos\phi \cos a \sec b \quad (2)$$
$$\cot B = \cot A \sin\phi' \csc\phi \quad (3)$$
$$\cot\theta = \cos b \tan A \quad (4)$$
$$\cot\theta' = \cos a \tan B \quad (5)$$

Check by sine formula

$$c = \phi + \phi' \quad (6)$$
$$C = \theta + \theta' \quad (7)$$

	° ′ ″	(1)	(2)	(3)	(4)	(5)
a	109 41 40	. . . . .	cos(−)	. . . . .	. . . . .	cos(−)
b	95 38 40	tan(−)	sec(−)	. . . . .	cos(−)	
A	132 36 30	cos(−)	. . . . .	cot(−)	tan(−)	
ϕ	81 41 38	tan 0.83567				
ϕ_1'	60 19 50	ϕ_2' −(60 19 50)	cos 9.69460	sin (±)		
B_1	128 55 40	B_2 51 04 20	. . . . .	cot (∓) 9.90725	. . . . .	tan (∓)
c_1	142 01 28	c_2 21 21 48				
θ	83 53 48		. . . . .	. . . . .	cot 9.02909	
θ_1'	67 21 07	θ_2' −(67 21 07)	. . . . .	. . . . .	. . . . .	cot (±) 9.62039
C_1	151 14 55	C_2 16 32 41				
		Check	sin A 9.86688	sin B_1 9.89094	sin C_1 9.68215	sin C_2 9.45448
			sin a 9.97382	sin b_1 9.99789	sin c_1 9.78910	sin c_2 9.56144
			9.89306	9.89305	9.89305	9.89304

EXERCISE LIV

1. Derive the formulae for the solution from Fig. 58.

2. In a spherical triangle given $a = 40° 06' 00''$, $b = 118° 22' 00''$, $A = 29° 43' 00''$; find B, c and C.

3. In a spherical triangle given $a = 140° 03' 00''$, $b = 131° 29' 00''$, $A = 127° 36' 00''$ find B, c and C.

4. In a spherical triangle given $a = 80° 05' 16''$, $b = 62° 04' 00''$, $A = 83° 34' 12''$; find B, C and c.

5. In a spherical triangle given $a = 70° 39' 50''$, $b = 45° 43' 36''$, $A = 31° 11' 20''$; find B, C and c.

6. In a spherical triangle given $a = 30° 38' 04''$, $b = 31° 29' 45''$, $A = 87° 53' 20''$; find B, c and C.

7. In a spherical triangle given $a = 31° 12' 30''$, $b = 61° 48' 20''$, $A = 35° 25' 25''$; find the remaining parts.

8. In a spherical triangle given $a = 52° 45' 20''$, $b = 71° 12' 40''$, $A = 46° 22' 10''$; find B, C, and c.

9. In a spherical triangle given $a = 70° 23' 00''$, $b = 105° 40' 00''$, $A = 71° 23' 00''$ find B, C, and c.

10. In a spherical triangle given $a = 110° 24' 40''$, $b = 76° 46' 36''$, $A = 120° 27' 20''$; find B, C and c.

11. If a spherical triangle is constructed from the given parts a, b, A, are there limitations on the values which may be assigned to a, b and A?

124. Case IV: Given A, B, a. This case can be solved by solving its polar triangle by the formulae given in Art. 123, or by dividing the triangle into two right triangles. In this case there may be two solutions. The formulae for the solution by the second method are as follows:

$\tan \phi = \cos B \cdot \tan a$, (1)

$\sin \phi' = \sin \phi \tan B \cdot \cot A$, (2)

$c = \phi + \phi'$ where $\phi' = \phi'_1$, and ϕ'_2, (3)

$\cos b = \cos a \cdot \cos \phi' \cdot \sec \phi$, (4)

$\cot \theta = \cos a \cdot \tan B$, (5)

$\cot \theta' = \cos b \cdot \tan A$, (6)

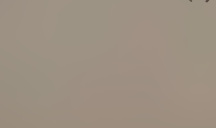


FIG. 59

$C = \theta + \theta'$. (7)

Check by the Sine formula.

There may be two solutions in this case but, since $0° < c < 180°$, values of ϕ' such that $\phi + \phi' > 180°$ or $\phi + \phi' < 0°$ must be discarded and thus the number of solutions is determined. It should be noted also that ϕ' may be negative.

EXERCISE LV

1. Derive the formulae just given from Fig. 59.

2. In a spherical triangle, given $A = 140°\ 47'\ 00''$, $B = 74°\ 32'\ 00''$, $a = 145°\ 47'\ 38''$; find b, c and C by solving the polar triangle by formulae of Art. 123.

3. In a spherical triangle, given $A = 139°\ 55'\ 00''$, $B = 61°\ 38'\ 00''$, $a = 150°\ 17'\ 00''$; find b, c and C by the formulae of Art. 124.

4. In a spherical triangle, given $B = 113°\ 30'\ 00''$, $b = 66°\ 44'\ 40''$, $A = 125°\ 31'\ 34''$; find a, c and C.

5. In a spherical triangle, given $B = 139°\ 52'\ 30''$, $A = 61°\ 32'\ 40''$, $b = 150°\ 16'\ 20''$; find a, c and C.

6. In a spherical triangle, given $A = 109°\ 41'\ 40''$, $B = 84°\ 21'\ 20''$, $a = 132°\ 36'\ 30''$; solve by the polar triangle.

7. In a spherical triangle, given $A = 148°\ 47'\ 30''$, $B = 118°\ 11'\ 40''$, $a = 144°\ 34'\ 35''$; find b, C and c.

8. In a spherical triangle, given $A = 70°$, $B = 120°$, $b = 80°$; find a, C and c.

9. In a spherical triangle, given $A = 115°\ 36'\ 45''$, $B = 80°\ 19'\ 12''$, $b = 84°\ 21'\ 56''$; find a, C and c.

10. In a spherical triangle, given $A = 61°\ 37'\ 53''$, $B = 139°\ 54'\ 34''$, $b = 150°\ 17'\ 26''$; find a, C and c.

125. Case V: Given a, b, c.— When the three sides are given the angles of the triangle can be found by the formula for the tangent of the half-angle (53) and (54), or by the haversine formula (50). If only one or two angles are to be found the haversine formula is the better one to use.

Example.—In a spherical triangle, given $a = 124°\ 12'\ 31''$, $b = 54°\ 18'\ 16''$, $c = 97°\ 12'\ 25''$; find the three angles.

(*a*)

$$P = \sqrt{\sin(s-a)\cdot\sin(s-b)\cdot\sin(s-c)\cdot\csc s}. \quad (1)$$

$$\tan\frac{A}{2} = P\cdot\csc(s-a). \quad (2)$$

$$\tan\frac{B}{2} = P\cdot\csc(s-b). \quad (3)$$

$$\tan\frac{C}{2} = P\cdot\csc(s-c). \quad (4)$$

Check by Sine Formula.

	°	′	″
a	124	12	31
b	54	18	16
c	97	12	25
2*s*	275	43	12

				(1)	(2)	(3)	(4)
s	137	51	36				
s−*a*	13	39	05	sin 9.37293	csc 0.62707		
s−*b*	83	33	20	sin 9.99725		csc 0.00275	
s−*c*	40	39	11	sin 9.81390			csc 0.18610
s	137	51	36	csc 0.17331			
				2)9.35739			
P				log 9.67870	log 9.67870	log 9.67870	log 9.67870
A	127	22	06, $\frac{A}{2}$	63 41 03	tan 0.30577		
B	51	18	10, $\frac{B}{2}$	25 39 05		tan 9.68145	
C	72	26	40, $\frac{C}{2}$	36 13 20			tan 9.86480

Check	sin *a* 9.91751	sin *b* 9.90962	sin *c* 9.99656
	sin *A* 9.90023	sin *B* 9.89235	sin *C* 9.97929
	0.01728	0.01727	0.01727

(*b*)

$$\text{hav } A = \sin(s-b)\cdot\sin(s-c)\cdot\csc b\cdot\csc c \quad (1)$$

$$\text{hav } B = \sin(s-a)\cdot\sin(s-c)\cdot\csc a\cdot\csc c \quad (2)$$

$$\text{hav } C = \sin(s-a)\cdot\sin(s-b)\cdot\csc a\cdot\csc b \quad (3)$$

Check by Sine formula.

	°	′	″	(1)	(2)	(3)
a	124	12	31		csc 0.08249	csc 0.08249
b	54	18	16	csc 0.09038		csc 0.09038
c	97	12	25	csc 0.00344	csc 0.00344	
2*s*	275	43	12			
s	137	51	36			
s−*a*	13	39	05		sin 9.37293	sin 9.37293
s−*b*	83	33	20	sin 9.99725		sin 9.99725
s−*c*	40	39	11	sin 9.81390	sin 9.81390	
s	137	51	36			
A	127	22	10	hav 9.90497		
B	51	18	09		hav 9.27276	
C	72	26	38			hav 9.54305

Check,	$\sin a$ 9.91751	$\sin b$ 9.90962	$\sin c$ 9.99656
	$\sin A$ 9.90022	$\sin B$ 9.89235	$\sin C$ 9.97929
	0.01729	0.01727	0.01727

In either solution find the sum $(s-a)+(s-b)+(s-c)=s$ as a check. Also as a check on the arithmetical work of the first solution we note that

$$\log\tan\frac{A}{2}+\log\tan\frac{B}{2}+\log\tan\frac{C}{2}+\operatorname{colog} P=\log\csc s.$$

126. Case VI: Given A, B, C.—When the three angles are given we can find the three sides by solving the polar triangle by the methods employed in Case V, for in the polar triangle $a'=180°-A$, $b'=180°-B$, $c'=180°-C$, and we can solve for A', B' and C' and then determine a, b and c. We can also solve by means of the formula for the tangent of a half side in Prob. 2 of Exercise LI or by the haversine formula of Prob. 6 of Exercise LI.

EXERCISE LVI

In the following five spherical triangles find the three angles; given

1. $a=20°\ 16'\ 38''$, $b=56°\ 19'\ 40''$, $c=66°\ 20'\ 42''$.

2. $a=25°\ 02'\ 30''$, $b=111°\ 30'\ 30''$, $c=98°\ 17'\ 34''$.

3. $a=69°\ 32'\ 20''$, $b=43°\ 55'\ 40''$, $c=54°\ 19'\ 30''$.

4. $a=67°\ 30'\ 30''$, $b=61°\ 23'\ 06''$, $c=72°\ 08'\ 22''$.

5. $a=70°\ 14'\ 20''$, $b=49°\ 24'\ 10''$, $c=38°\ 46'\ 10''$.

6. In a spherical triangle, given $A=29°\ 42'\ 34''$, $B=42°\ 37'\ 18''$, $C=160°\ 01'\ 22''$, find the three sides by solving the polar triangle.

7. Solve the triangle of Ex. 6 by the formula of Prob. 2, Exercise LI.

8. Solve the triangle of Ex. 6 by using Prob. 6, Exercise LI.

Find the sides of the following spherical triangles; given

9. $A=136°\ 59'\ 34''$, $B=32°\ 37'\ 00''$, $C=37°\ 49'\ 04''$.

10. $A = 58° 28' 36''$, $B = 51° 15' 22''$, $C = 90° 36' 52''$.

11. $A = 105° 43' 32''$, $B = 75° 39' 23''$, $C = 68° 37' 25''$.

12. If a spherical triangle is constructed, given the three sides, what restrictions are there as to the values that may be assigned to the three sides?

13. If a spherical triangle is constructed, given the three angles, what restrictions are there as to the values which may be assigned to the three angles?

EXERCISE LVII

Solve the following spherical triangles given:

1. $A = 53° 37' 22''$, $C = 118° 29' 48''$, $c = 123° 48' 53''$.

2. $a = 69° 35' 20''$, $b = 103° 13' 24''$, $A = 59° 32' 40''$.

3. $B = 101° 52' 07''$; $c = 74° 50' 29''$, $A = 31° 09' 44''$.

4. $A = 91° 36' 00''$, $B = 123° 12' 00''$, $c = 51° 44' 00''$.

5. $B = 97° 59' 30''$, $a = 117° 07' 36''$, $c = 147° 02' 33''$.

6. $b = 80° 38' 28''$, $A = 60° 17' 38''$, $c = 104° 28' 13''$.

7. $a = 80° 37' 23''$, $b = 65° 18' 21''$, $c = 57° 44' 54''$.

8. $A = 63° 40' 20''$, $B = 89° 28' 22''$, $C = 72° 41' 50''$.

9. $a = 83° 10' 30''$, $C = 124° 30' 00''$, $b = 31° 54' 21''$.

10. $a = 115° 10' 00''$, $b = 32° 16' 00''$, $c = 88° 51' 23''$.

CHAPTER XIV

APPLICATIONS: THE TERRESTRIAL AND ASTRONOMICAL TRIANGLES

127. The Terrestrial Sphere.—In finding the great circle distances between points on the surface of the earth we consider the earth to be a sphere of about 3959 statute miles radius. To be more exact the radius is such that the length of 1′ of arc of a great circle is 6080.27 ft. = 1 nautical mile.

The diameter about which the earth revolves is called the *axis of the earth.*

The extremities of the axis are called the *north* and *south poles.*

The *equator* is a great circle whose plane is perpendicular to the axis of the earth.

Meridians are great circles passing through the poles of the earth; thus all meridians are perpendicular to the equator.

The *latitude* (denoted by L) of a point on the terrestrial sphere is its angular distance north or south of the equator. Latitude is measured on the meridian passing through the point and is called *north* or *south* according as it is measured north or south of the equator.

The *longitude* (denoted by λ) of a point on the terrestrial sphere is the angle at the poles between the meridian passing through the point and a fixed meridian. The fixed meridian, for example the meridian of Greenwich or of Washington, is taken as the zero meridian; i.e., $\lambda = 0°$. Longitude is taken as *positive* to the *west* and *negative* to the *east* of the zero meridian from 0° to 180°, or, in time units, from 0 hours to 12 hours. (Since the earth makes a complete revolution in 24 hours, it turns through 15° in one hour.)

The position of a point on the terrestrial sphere is determined by giving its latitude and longitude. These quantities are called the spherical coördinates of a point on the terrestrial sphere. Thus, to locate Annapolis, Md., we say $L = 38° 58' 53''$ N, $\lambda = 76° 29' 08''$ W. If the latitude and longitude are given to seconds they determine a position within a radius of about 100 ft. Hence when a place is located with this degree of accuracy some particular position has been chosen as a point of observation.

128. The Terrestrial Triangle.—If two points M_1, (spherical coördinates L_1, λ_1) and M_2 (L_2, λ_2) are given on the surface of

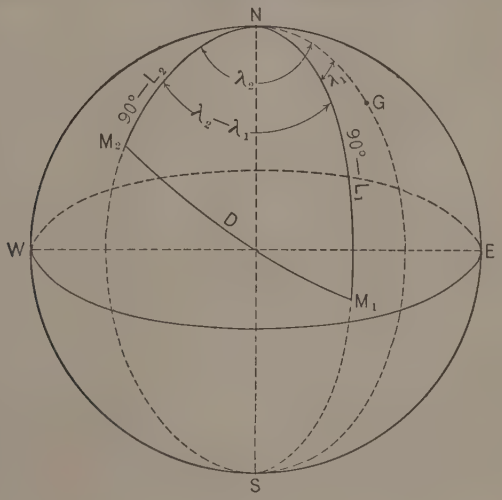


Fig. 60

the earth, the arc of the great circle joining them and the meridians of the two places form a triangle. This is known as the terrestrial triangle. Its sides are $90° - L_1$, $90° - L_2$, arc M_1M_2, and one angle is $\lambda_2 - \lambda_1$, as shown in Fig. 60. Arc M_1M_2, denoted by D, is the great circle distance between

the two points. We shall let $C = \angle NM_1M_2$ denote the course from M_1 to M_2 (C_N = compass course), and $DL_0 = \lambda_2 - \lambda_1$ denote the difference of longitude.

We thus have given two sides and the included angle. (See Art. 119.) If we drop a perpendicular from M_2 upon the side $90° - L_1$ as in Fig. 61, then by Napier's Rules:

$$\sin \overline{DL_0} = \tan \phi \tan (90° - \overline{L_2}), \text{ or } \tan \phi = \cos DL_0 \cot L_2 \quad (1)$$

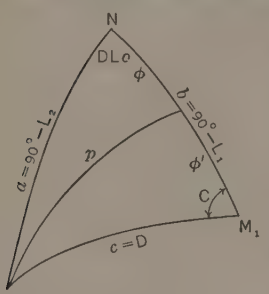


Fig. 61.

$$\sin (\overline{90° - L_2}) = \cos p \cos \phi$$

and

$$\sin \overline{D} = \cos p \cos \phi'$$

whence

$$\cos D = \sec \phi \cos \phi' \sin L_2$$

or

$$\cos D = \sec \phi \sin (L_1 + \phi) \sin L_2 \quad (2)$$

again

$$\sin \phi = \tan \overline{DL_0} \tan p \text{ and } \sin \phi' = \tan \overline{C} \tan p$$

whence
$$\cot C = \cos (L_1 + \phi) \csc \phi \cot DL_0 \quad (3)$$

where
$$\phi' = 90° - L_1 - \phi \quad (4)$$

Example.—Find the great circle course and distance between Vancouver, $L = 49°\ 16'\ N$, $\lambda = 8^h\ 12^m\ 44^s\ W$, and Yokohama, $L = 35°\ 30'\ N$, $\lambda = 9^h\ 18^m\ 40^s\ E$.

Since λ_1 is west and λ_2 is east $(\lambda_2 - \lambda_1) = 8^h\ 12^m\ 44^s + 9^h\ 18^m\ 40^s = 17^h\ 31^m\ 24^s$. Since this is greater than 12 hours we use $24^h - 17^h\ 31^m\ 24^s = 6^h\ 28^m\ 36^s$.

		(1)		(2)		(3)	
$DL_0 = 6^h\ 28^m\ 36^s =$	97 09 00	cos(−)	9.09506			cot(−)	9.09845
$L_2 =$	35 30 00	cot	0.14673	sin	9.76395		
$L_1 =$	49 16 00						
$\phi =$	170 06 06	tan(−)	9.24179	sec(−)	0.00652	csc	0.76473
$L_1 + \phi =$	219 22 06			sin(−)	9.80230	cos(−)	9.88823
$D = 4082.5$ miles =	68 02 34			cos	9.57277		
$C =$	60 34 10					cot	9.75141

C_N 299° 25′ 50″.

The "course and distance problem" can be solved by formulae obtained from (47) and (48).

Applying the haversine formula (47) to Fig. 61, we have

$$\text{hav } D = \text{hav } (90° - L_1 - 90° + L_2) + \text{hav } DL_0 \sin (90° - L_2) \sin (90° - L_1)$$

or

$$\text{hav } D = \text{hav } (L_2 - L_1) + \text{hav } DL_0{}^* \cos L_2 \cos L_1.$$

Applying the Sine Law (48), we have

$$\frac{\sin C}{\sin DL_0} = \frac{\sin (90° - L_2)}{\sin D},$$

or

$$\sin C = \cos L_2 \csc D \sin DL_0.$$

However, there is the ambiguity as to the quadrant of C; if there is no doubt as to the quadrant of C these formulae offer a ready solution of the problem.

EXERCISE LVIII

1. Find the distance in geographic miles from Annapolis, $L = 38° 59' 53'' N$, $\lambda = 76° 29' 08'' W$, to Greenwich, England, $L = 51° 28' 38'' N$, $\lambda = 0° 0' 0''$.

2. Compute the initial course and distance sailed for the great circle, from Cape Inscription, $L = 25° 29' S$, $\lambda = 112° 57' E$, to the Cape of Good Hope, $L = 34° 22' S$, $\lambda = 18° 30' E$.

3. Use the formulae for hav D and sin C, given above to solve the illustrative example.

4. Use the formulae of Ex. 3 to solve Ex. 2.

* In Table 45, *Bowditch Useful Tables*, the value of the angle is given in hours, minutes and seconds in light type—the hours and minutes at the top of the column of haversines, and the seconds at the extreme left, for angles between 0^h and 12^h, and the hours and minutes at the bottom, and the seconds at the right, for angles between 12^h and 24^h. Thus, the value of log hav DL_0 can be read directly from this table without first expressing DL_0 in sexagesimal measure.

5. Compute the initial course and great circle distance from San Francisco, $L=37°\ 50'\ N$, $\lambda=122°\ 30'\ W$, to Yokohama, $L=35°\ 30'\ N$, $\lambda=140°\ 00'\ E$.

6. Assuming that the continent of Asia forms approximately a great spherical triangle which has its vertices at East Cape, $L_1=66°\ 02'\ N$, $\lambda_1=169°\ 32'\ 30''\ W$, Perim Island, $L_2=12°\ 39'\ N$, $\lambda_2=43°\ 25'\ 35''\ E$, and Singapore Strait, $L_3=1°\ 20'\ N$, $\lambda_3=104°\ 24'\ E$, find approximately the area of the continent of Asia.

7. A vessel sails a great circle course from San Francisco, $L=37°\ 50'\ N$, $\lambda=8^h\ 10^m\ 45^s\ W$, to Sydney, $L=33°\ 52'\ S$, $\lambda=10^h\ 04^m\ 00^s\ E$; find the distance and initial course. If she makes 15 knots, what course is she sailing exactly 5 days after leaving San Francisco? (Hint: The great circle course is constantly changing, since by definition the course is the angle between the great circle path and the meridian of the ship.)

Find initial course and great circle distance from the point determined by L_1, λ_1 to the point determined by L_2, λ_2 if

	L_1	L_2	λ_1	λ_2
8.	$21°25'35''\ N$	$28°42'00''\ S$	$55°52'00''\ E$	$121°21'00''\ E$
9.	$26°24'19''\ N$	$14°42'00''\ N$	$105°40'00''\ E$	$37°18'00''\ E$
10.	$31°48'00''\ S$	$30°41'00''\ N$	$20°37'00''\ W$	$42°51'00''\ E$
11.	$21°39'35''\ N$	$37°41'45''\ N$	$38°42'00''\ E$	$78°30'20''\ W$

12. From the haversine laws (47) and (50) derive the formulae $\text{hav}\ D=\sin\ \text{co}\ L_1 \sin\ \text{co}\ L_2\ \text{hav}\ DL_0+\text{hav}\ (L_1-L_2)$ and $\text{hav}\ C=\text{hav}\ \phi\ \csc\ D\ \csc\ \text{co}\ L_1$, $\text{hav}\ \phi=\text{hav}\ \text{co}\ L_2 -\text{hav}\ (D-\text{co}\ L_1)$, where co $L=$ colatitude, $DL_0=$ difference of longitude, $C=$ course.

13. Use the formulae given in Ex. 12 to solve Ex. 5.

14. Use the formulae given in Ex. 12 to solve Ex. 8.

15. Let V be the vertex, or point of highest latitude reached on the great circle track from M_1 to M_2. If we draw the meridian of V, the angle at V is a right angle. By Napier's Rules derive the formulae

$$\sin \text{co } L_v = \sin \text{co } L_1 \sin C$$

and

$$\tan DLo_v = \sec \text{co } L_1 \cot C,$$

also

$$\cot L_m = \tan \text{co } L_v \sec \theta$$

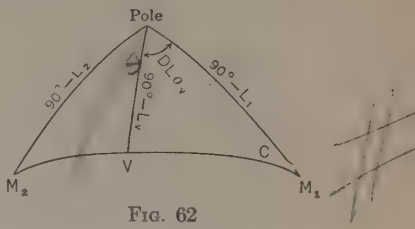


FIG. 62

for a point M on the great circle whose longitude is θ° different from V.

16. Compute the great circle distance and initial course from Brisbane, Lat. 27° 27′ 32″ S, Long. 153° 01′ 48″ E, to Acapulco, Lat. 16° 49′ 10″ N, Long. 99° 55′ 50″ W. Compute also the latitude and longitude of the vertex.

129. The Celestial Sphere.—An observer upon the surface of the earth appears to view the heavenly bodies as if they were situated upon the surface of a vast hollow sphere, of which his eye is the center. The *Celestial Sphere* is an imaginary spherical surface of infinite radius, the eye of the observer being at the center. The positions of the heavenly bodies are projected upon this sphere at points where lines joining them with the center pierce the surface of the sphere. Since, however, the center of the earth should be the point from which all angular distances are measured, the observer, by transferring himself there, will find projected on the celestial sphere, not only the heavenly bodies, but the imaginary points and circles of the earth's surface. The actual position of the observer will be projected in a point called the *zenith*; the meridians, equator and all other lines and points may also be projected.

An observer on the earth's surface is constantly changing his position with relation to the celestial bodies projected on the sphere, thus giving to the latter an apparent motion. The changes produced by the rotation of the earth about its axis are different for different observers at different points upon the earth and, therefore, depend upon the latitude and longitude of the observer.

The *Zenith* (Z) of an observer on the earth's surface is the point of the celestial sphere vertically overhead. The *Nadir* (Na) is the point vertically beneath.

The *Celestial Horizon* is the great circle of the celestial sphere formed by passing a plane through the center of the earth at right angles to the line joining the zenith and nadir.

The *Equinoctial*, or *Celestial Equator*, is the great circle formed by extending the plane of the earth's equator until it intersects the celestial sphere. The equinoctial intersects the horizon in E and W, its east and west points. The *Poles of the Sphere* are the points towards which the earth's axis is directed.

Hour Circles, or *Celestial Meridians*, are great circles of the celestial sphere passing through the poles; they may be formed by extending the planes of the respective terrestrial meridians until they intersect the celestial sphere. Thus the Celestial Meridian of a place on the earth is the extension of the terrestrial meridian of that place.

Vertical Circles are great circles of the celestial sphere which pass through the zenith and nadir.

The *Prime Vertical* is the vertical circle through the *east* and *west* points of the horizon. It is perpendicular to the horizon and the meridian of the observer.

The *Declination* (d) of any point in the celestial sphere is its angular distance from the equinoctial, measured upon the hour or declination circle which passes through that point; it is designated as north or south according to the direction of the point from the equinoctial; it is customary to regard *north*

declination as *positive* (+) and *south* declination as *negative* (−). Declination upon the celestial sphere corresponds to latitude upon the earth.

The *Polar Distance* (p) of any point is its angular distance from the pole measured upon the hour or declination circle passing through the point. It must therefore equal 90° minus the declination, if measured from the pole of the same name as the declination, or 90° plus the declination if measured from the pole of opposite name.

The *Altitude* (h) of any point in the celestial sphere is its angular distance from the horizon, measured upon the vertical circle passing through the point; it is regarded as positive when the body is on the same side of the horizon as the zenith.

The *Zenith Distance* (z) of any point is its angular distance from the zenith, measured upon the vertical circle passing through the point; the zenith distance of any point which is above the horizon of an observer must therefore equal 90° minus the altitude.

The *Hour Angle* (t) of any point is the angle at the pole between the meridian of the observer and the hour circle passing through the point; it may also be regarded as the arc of the equinoctial intercepted between those circles. It is measured toward the west as a positive direction through the twenty-four hours, or 360°.

The *Azimuth* (Z) of a point in the celestial sphere is the angle at the zenith between the meridian of the observer and the vertical circle passing through the point. It is measured from either the north or south point of the horizon to the east or west through 180°.

The *Amplitude* of a point is the angle at the zenith between the prime vertical and the vertical circle of the point; it is measured from the east or the west point of the horizon through 90°, as E 40° N. The amplitude is only used with reference to points in the horizon.

Figures 63 and 64 are intended to show the earth in relation to a sphere, similar to the celestial sphere, to which the celestial sphere may be considered to have shrunk. In Fig. 63 the observer's sensible horizon is a circle with center at O and in a plane parallel to the celestial horizon $NWSE$ through the center of the earth; also the azimuth, Z_N, is measured from the north to the east through 360° as on the compass, though

Fig. 63. Horizon System

in existing azimuth tables the azimuth (Z) is still reckoned from the elevated pole toward the east or west.

The *Ecliptic* is the great circle representing the path in which by reason of the annual revolution of the earth the sun appears to move in the celestial sphere; the plane of the ecliptic is inclined to that of the equinoctial at an angle of 23° 27′.

Figure 65 represents a sphere, similar to the celestial sphere, with the sun and orbit of the earth (magnified) at its center. It should be remembered that the finite dimensions of the orbit of the earth are negligible in comparison with the infinite radius of the celestial sphere, and thus the earth may always be considered as at the center of the celestial sphere. As the

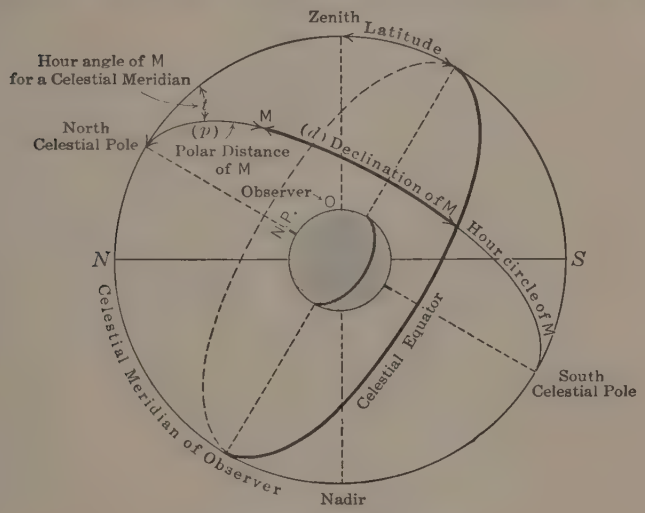


Fig. 64. Equinoctial System

earth makes its annual journey around the sun its axis keeps a fixed direction and is inclined at about 67° to the plane of its orbit. When the earth has the position E_1 the sun is projected on the celestial equator into the point *V.E.* or ♈ (Vernal Equinox). When the earth is at E_1 its axis is not inclined at all toward the sun. When the earth has the position E_2 (where the northern half of the earth's axis has its greatest inclination toward the sun) the sun appears to be at the point *S.S.* (Sum-

mer Solstice), about 23° north of the celestial equator. When the earth is at E_3 the sun appears to have the position $A.E.$ (Autumnal Equinox), and when the earth has the position E_4 the sun appears at $W.S.$ (Winter Solstice), about 23° south of the celestial equator. The ecliptic is thus the path in which the sun appears to move on the celestial sphere due to the annual revolution of the earth.

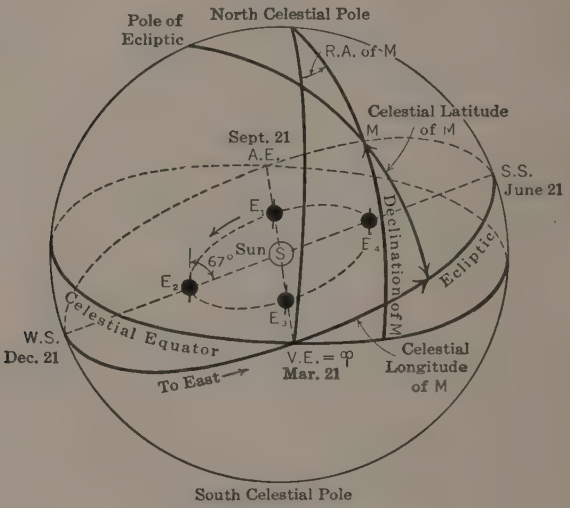


Fig. 65. Ecliptic System

M represents the position of a star at some instant and its coördinates in the ecliptic system, celestial longitude and celestial latitude, are shown.

The *Equinoxes* are those points at which the ecliptic and the equinoctial intersect, and when the sun occupies either of these positions the days and nights are of equal length throughout the earth. The *Vernal Equinox* is that one at which the sun appears to an observer on the earth when passing from *southern*

to northern declination, and the *Autumnal Equinox* that one at which it appears when passing from *northern to southern* declination. The Vernal Equinox is also designated as the First Point of Aries, and is used as an origin for reckoning right ascension.

The *Right Ascension* of a point is the angle at the pole between the hour circle of the point and that of the Vernal

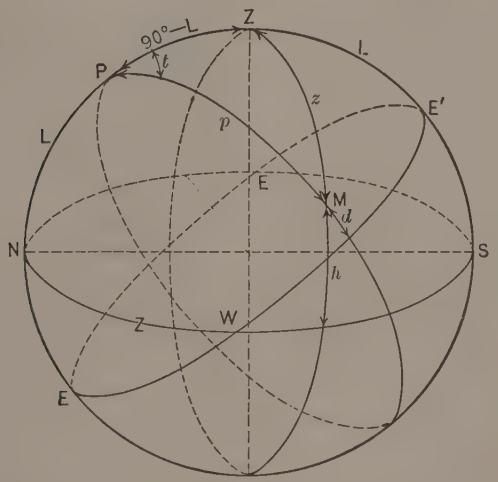


Fig. 66

Equinox; it may also be regarded as the arc of the equinoctial intercepted between these circles. It is measured from the Vernal Equinox to the *eastward* as a *positive direction* through twenty-four hours or 360°.

The passage of a celestial body across the meridian is called its *transit* or *culmination*. When the body is within the circle of perpetual apparition, both transits occur above the horizon. These are called the upper and lower transits. For all bodies

outside this circle, the upper transit occurs above the horizon, the lower below it. For all bodies whatever, the upper transit occurs when the body crosses the upper branch of the meridian and the lower transit when it crosses the lower branch.

The position of a body on the celestial sphere with reference to an observer is determined if the altitude (or zenith distance)

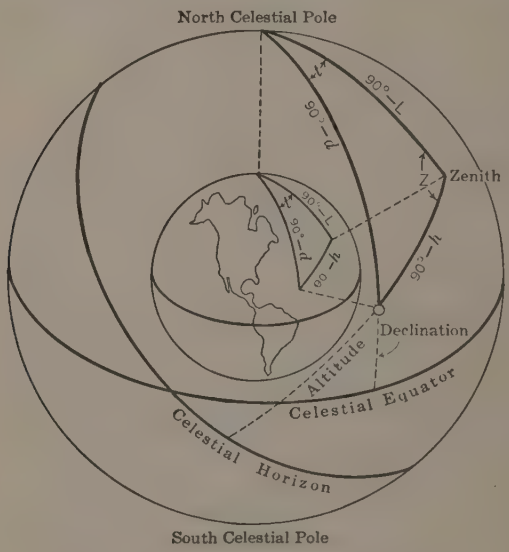


FIG. 67. THE ASTRONOMICAL TRIANGLE

and azimuth (see Fig. 63), or, (2) the declination (or polar distance) and hour angle (see Fig. 64), or, (3) the right ascension and declination (see Fig. 65) are known. Any pair of these three pairs of quantities thus constitute the spherical coördinates of a point on the sphere.

130. The Astronomical Triangle.—The triangle formed by the meridian of the observer, the hour circle through any

point M, and the vertical circle through M is called the astronomical triangle. Thus the vertices of the triangle are P, Z, and M; the sides are $90°-L$, the co-latitude; $p=90°-d$, the polar distance; and $z=90°-h$, the zenith distance. The angle at P is t, the hour angle; the angle at Z is the azimuth, Z; and the angle at M is called the position angle. (See Fig. 66.)

Figure 67 shows the relation of the astronomical triangle to a similar triangle drawn on the earth, such that the north pole and the place of the observer correspond respectively to the north celestial pole and the zenith in the astronomical triangle. The homologous parts of these triangles have the same measure in degrees, since they depend only on the angles of the trihedral at the center of the sphere and not on the radius of the sphere. We first associate latitude with the earth, whereas we associate altitude and declination with the celestial sphere, but these three quantities determine the respective sides of either triangle. This relationship is the basis of celestial navigation.

131. To Find the Latitude of a Place on the Earth.—Since Z is on a diameter of the earth produced through the position of the observer, and the celestial equator is in the plane of the earth's equator, it follows that the declination of Z gives the angular distance from the equator, that is, the latitude, of the position of the observer. Also since P is 90° from the equator, the angular distance of P above the horizon, the altitude of P, is equal to the latitude of the place of the observer. Thus, the latitude of a place on the earth is equal to the declination of the zenith, or the altitude of the elevated or visible pole. Let Fig. 68 represent a projec-

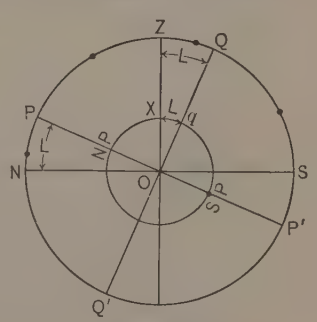


Fig. 68

tion of the celestial sphere on the plane of the meridian NZS; the larger circle represents the celestial sphere and the smaller one represents the earth. O is the center of both spheres; NS is the horizon; P and P' are the poles of the celestial sphere; QOQ', the equator; Z, the zenith of the observer; NP and SP are the north and south poles, respectively. The latitude of X is $qX = QZ = \text{arc } NP$.

If the zenith distance of some heavenly body, when on the meridian of the observer, and the declination of that body be known, the latitude of that place can be found. The altitude of the sun or some other heavenly body when just on the meridian can be determined by observation and proper corrections made to give the true altitude above the horizon. The declination of the same body can be found from the American Nautical Almanac, in which the declinations of the brighter celestial bodies are tabulated. Then, if M be the position of such a body, as in Fig. 69, we have

$$L = QZ = QM + MZ.$$

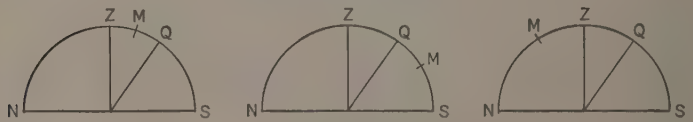


FIG. 69

The student will find it best to make a construction in each case. First draw the meridian of the place, locate M by measuring h from the north horizon if it bears north, and from the south horizon if M bears south; then measure d from M in a direction opposite to its name, if M is in upper culmination, thus locating Q. If the body is in lower culmination, use $(180° - d)$ in place of d, again measuring the direction opposite to the name of d. Then L is the declination of Z in both magnitude and name, i.e., $L = QZ$ and is marked N when Z is north of Q, and marked S when Z is south of Q.

Example.—Find L if $d = 20° 25'$ S, $h = 40° 30'$ bearing south, body in upper transit.

By our rule from Fig. 70

$$L = QZ = (90° - 60° 55')\ N$$
$$= 29° 05'\ N.$$

We can also obtain the latitude by measuring the altitude of the elevated or visible pole. Thus in Fig. 70,

$$L = NP_N = 29° 05' N.$$

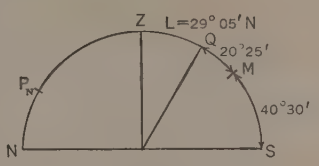


FIG. 70

EXERCISE LIX

From the meridian altitude (h), declination (d), and bearing of the observed body find the latitude of the observer in each of the following cases.

(a) Assume in each case that the body is in upper culmination.

	d	h	*bearing*
1.	27° 28′ S	52° 22′	S
2.	33° 50′ S	67° 30′	N
3.	16° 53′ N	37° 18′	S
4.	22° 16′ N	81° 52′	N
5.	18° 28′ S	79° 14′	S
6.	25° 13′ S	62° 47′	N
7.	20° 10′ N	32° 16′	S
8.	81° 48′ N	45° 26′	S
9.	46° 25′ N	22° 18′	N
10.	44° 26′ S	38° 06′	S
11.	10° 52′ S	43° 27′	S
12.	23° 13′ N	75° 40′	S
13.	61° 42′ N	52° 35′	N
14.	55° 00′ N	75° 00′	N

(*b*) Assume the body to be in lower culmination

	d	*h*	*bearing*
15.	44° 26′ *S*	38° 06′	*S*
16.	62° 00′ *N*	50° 00′	*N*
17.	73° 16′ *N*	28° 48′	*N*
18.	46° 25′ *N*	22° 18′	*N*

132. Given L, d, h: To Find t and Z.—This problem is called the *time sight* and is one of the most important in nautical astronomy because of its almost daily application by the navigator to determine the longitude of his ship. The altitude of a celestial body is measured with a sextant, its declination is taken from the tables of the Nautical Almanac, and the latitude of the observer is assumed to be known. The determination of t and Z then reduces to the solution of a spherical triangle of the type of Case V (Art. 125), where, as in Fig. 71,

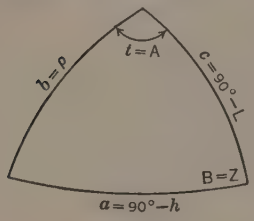


FIG. 71

$$a=90°-h,\ b=p,\ c=90°-L,\ A=t \text{ and } B=Z$$

Then $$2S=180°+p-h-L=180°-2S+2p,$$

where $2S=h+p+L$, (Note that p, and not d, is here.)

and
$$S=90°-(S-p),$$
$$S-a=h-(S-p)=S-L,$$
$$S-b=90°-(S-p)-p=90°-S,$$
$$S-c=90°-S+p-90°+L=S-h.$$

Substituting these values in (50)

$$\text{hav } t=\frac{\sin (90°-S)\sin (S-h)}{\sin p \sin (90°-L)}$$

or,
$$\text{hav } t=\cos S \sin (S-h) \sec L \csc p;$$

and in the formula for hav B, corresponding to (50)

$$\text{hav } Z = \frac{\sin (S-h) \sin (S-L)}{\sin (90°-h) \sin (90°+L)};$$

or $$\text{hav } Z = \sin (S-h) \sin (S-L) \sec h \sec L.$$

When the sun is the body observed a simple relation exists between the hour angle and the local apparent time. Since the sun appears to move from east to west at the hourly rate of 15°, and crosses the meridian of the place at apparent noon, we have the following equation: local apparent time in hours $= 12 - t/15$ A.M., or $t/15$ P.M. (where t is in sexagesimal measure), according as the sun is east or west of the meridian.

Illustration.—Given $L = 30° N$, $h = 30°$ bearing E, and $d = 15° S$, find t and Z.

	°	′	″				
$h =$	30	00	00			sec	0.06247
$L =$	30	00	00	sec	0.06247	sec	0.06247
$p =$	105	00	00	csc	0.01506		
$2S =$	165	00	00				
$S =$	82	30	00	cos	9.11570		
$S-h =$	52	30	00	sin	9.89947	sin	9.89947
$S-L =$	52	30	00			sin	9.89947
$t =$	41	12	02[1]	hav	9.09270		
$t =$	2^h	44^m	48^s				
$Z = N$	132	43	20 E			hav	9.92388

Z is marked NE because the Astronomical triangle as used contained the North Pole and the body bears east; had we used the South Pole we should have $Z = S\ 47°\ 16'\ 40'' E$.

EXERCISE LX

1. Given $L = 45° N$, $d = 22°\ 30' N$, $h = 30°$ bearing W, compute t and Z, expressing t in time units.

[1] Bowditch, Table 45, gives the hours, minutes, and seconds directly.

2. Draw and letter the astronomical triangle, marking the values of the given parts on the figure, given $L = 30° 25' N$, $d = 14° 20' S$, $h = 27° 35' W$; then compute t and Z.

3. Given $L = 48° 10' N$, $d = 5° 20' S$, $h = 25° 15' W$, compute t and Z.

4. Given $L = 31° 50' N$, $d = 3° 04' S$, $h = 14° 48' E$, compute t and Z, expressing t in time units.

5. Given $L = 30° N$, $d = 15° S$, $h = 15° E$, compute t and Z.

6. Given $h = 30° 10' 30''$ bearing E, $d = 22° 31' 20'' N$, $L = 44° 40' N$, compute t and Z.

7. Find the local apparent time of sunrise and sunset at

(*a*) Cape Nome, $L = 64° 26' N$, if $d = 23° 27' N$.

(*b*) London, $L = 51° 29' N$, if $d = 13° 17' N$.

(*c*) Panama, $L = 8° 57' N$, if $d = 18° 29' N$.

(*d*) Naval Academy, Annapolis, $L = 38° 58' 53''$, if $d = 22° 29' N$.

8. Find the length of the shortest day; of the longest day; at Petrograd ($L = 59° 56' 30'' N$, $\lambda = 30° 19' 22'' E$).

9. Find the length of the shortest day; of the longest day; at Valparaiso ($L = 33° 01' 08'' S$, $\lambda = 71° 38' 52'' W$).

10. If twilight continues until the sun is 18° below the horizon, find the length of dawn, dark night, bright day, and twilight in New York (Lat. $40° 43' N$): (*a*) at the Winter Solstice; (*b*) at the Summer Solstice; (*c*) at the Equinoxes.

11. Given $L = 34° 29' S$, $d = 12° N$, $h = 30° W$, compute t and Z.

12. Given $L = 62° 40' S$, $d = 22° 29' S$, $h = 44° 57' E$, compute t and Z.

13. Given $L = 16° 14' S$, $d = 30° 18' N$, $h = 15° 32' W$, compute t and Z.

133. Given t, d, L: To find h and Z.—In this case we are given two sides and the included angle, or Case I. (Art 119.) Dropping the perpendicular from M, we have in Fig. 72

$$a=90°-L,\ b=90°-d,\ C=t,$$
$$c=90°-h,\ \text{and}\ B=Z,$$

and by Napier's Rules

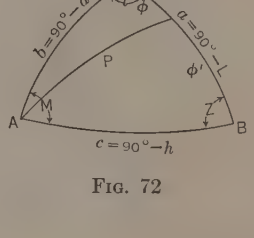


FIG. 72

$$\sin \overline{t} = \tan \phi \tan (\overline{90°-d})$$

or
$$\tan \phi = \cos t \cot d, \qquad (1)$$

$$\sin (\overline{90°-d}) = \cos \phi \cos p$$

and
$$\sin (\overline{90°-h}) = \cos \phi' \cos p$$

whence
$$\sin h = \sec \phi \sin (L+\phi) \sin d \qquad (2)$$

$$\sin \phi = \tan p \tan \overline{t} \quad \text{and} \quad \sin \phi' = \tan p \tan \overline{Z}$$

whence
$$\cot Z = \cot t \csc \phi \cos (L+\phi) \qquad (3)$$

$$\phi' = 90°-L-\phi. \qquad (4)$$

When d has a different name from L, d is negative.

An illustration of the case follows:

	° ′ ″	(1)	,	(2)		(3)	
$L=$	42 00 00 N						
$t=$	55 00 00 E	cos 9.75859				cot	9.84523
$d=$	−15 00 00 S	cot(−) 0.57195		sin(−) 9.41300			
$\phi=$	115 02 24	tan(−) 0.33054		sec(−) 0.37340		csc	0.04286
$\phi+L=$	157 02 24			sin 9.59116		cos(−)	9.96415
$h=$	13 48 01			sin 9.37756			
$Z=N$	125 26 09 E					cot(−)	9.85224

134. In the practice of navigation the Method of Saint Hilaire is generally used rather than the solution given in Art. 133. In this method the value of Z is taken from Azimuth Tables, and $z=90°-h$ is computed by the haversine formula (47). The navigator compares the computed value of h with the observed value, and from the difference between the two determines the correction to be applied to the assumed position

of his ship. In the formula (47) let $a=z$, $b=90°-d$, $c=90°-L$, and $A=t$, then

$$\text{hav } z=\text{hav } (L-d)+\cos L\cdot\cos d\cdot\text{hav } t.$$

Illustration.—Given $t=55°$ E, $L=42°$ N, $d=15°$ S, to find h.

	°	′	″				
$t=$	55	00	00			hav	9.32881
$L=$	42	00	00			cos	9.87107
$d=-$	15	00	00			cos	9.98494
				N hav	0.15304	hav	9.18482
$L-d=$	57	00	00	N hav	0.22768		
$z=$	76	11	55	N hav	0.38072		
$h=$	13	48	05				

EXERCISE LXI

1. Given $t=50°\ 20'$ W, $d=20°\ 30'$ N, $L=29°\ 40'$ N, compute h and Z.

2. Given $t=25°$ W, $d=10°$ S, $L=40°$ N, compute h and Z.

3. Given $t=45°$ W, $d=15°$ S, $L=30°$ N, compute h and Z.

4. Given $t=32°\ 20'$ E, $d=15°\ 14'$ N, $L=46°\ 18'$ N, compute h and Z.

5. Given $L=44°\ 45'$ N, $d=59°\ 56'$ N, $t=60°\ 32'$ E, compute h and Z.

6. Given $L=30'$ N, $d=4°$ N, $t=63°$ W, compute h and Z.

7. Given $d=30°\ 21'\ 14''$ S, $L=29°\ 17'\ 24''$ N, $t=67°\ 42'\ 12''$ E, compute h and Z.

8. Given $t=40°$ W, $d=12°$ N, $L=34°\ 28'\ 40''$ S, compute h and Z.

9. Given $t=25°$ E, $d=10°$ S, $L=18°\ 57'\ 16''$ S, compute h and Z.

10. Given $t=30°\ 28'$ W, $d=18°\ 25'$ S, $L=69°\ 16'\ 07''$ S, compute h and Z.

11. From Napier's Analogies (Art. 115) derive the formulae

$$\tan \tfrac{1}{2}(Z-M) = \cot \tfrac{1}{2}t \sin \tfrac{1}{2}(L-d) \sec \tfrac{1}{2}(L+d),$$
$$\tan \tfrac{1}{2}(Z+M) = \cot \tfrac{1}{2}t \cos \tfrac{1}{2}(L-d) \csc \tfrac{1}{2}(L+d).$$

12. Use the formulae of Prob. 11 to compute Z from the data of Prob. 1.

13. Use the formulae of Prob. 11 to compute Z from the data of Prob. 7.

EXERCISE LXII

1. A star having the known declination d is observed to have the altitude h when crossing the Prime Vertical. Using Napier's Rules, derive formulas from the right triangle PZM for t and L.

2. A star with declination 49° 58′ N makes its prime vertical transit in altitude 62° 19′. Find its hour-angle and the observer's latitude.

3. On a certain day it happens that the effect of the semi-diameter of the moon and the refraction are offset by the parallax, so that the moon rises when its altitude is zero. Given the moon's declination and the observer's latitude, derive formulas from Napier's Rules for the right triangle PNM to determine the hour-angle and amplitude of the moon's rising.

4. Find the hour-angle and amplitude of moon-rise at Annapolis, Lat. 38° 59′ N, on a day when the moon's declination is 25° 28′ N, and the corrections neutralize as in Prob. 3.

5. Declination of a star 37° 44′ S, altitude at prime vertical transit 48° 19′; find its hour-angle and observer's latitude.

6. Declination of moon 15° 36′ S, moon-rise for $h = 0°$, latitude of observer 38° 59′ N; find hour-angle and amplitude of moon-rise.

7. The northern vertex of a great circle track is in Lat. 68° 27′ N, Long. 20° 23′ W. A ship sails on this track from Long. 37° 18′ W to Long. 26° 28′ W. Find its latitude, course, and distance from the vertex in each position.

8. The southern vertex of a great circle track is in Long. 123° 13′ 30″ E, Lat. 54° 45′ S. A ship sails 70 nautical miles along this track toward the vertex from a position in Lat. 10° S. What position does it reach? What is its course at each position?

9. The northern vertex of a great circle track is in Long. 163° 33′ W, Lat. 46° 07′ N. How far must a ship sail along this track to change its course from 256° 30′ to 255° 30′? What are its two corresponding positions?

10. On the track in Prob. 9, how far must a ship sail to change its course from 270° 30′ to 269° 30′? From 226° 30′ to 225° 30′?

11. Using the formulae of Prob. 12, Exercise LVIII, find the initial course and distance by great circle from San Francisco, 38° 28′ 24″ N, 123° 22′ 54″ W, to Manila, 14° 35′ 48″ N, 120° 57′ 18″ E.

12. Find the northern vertex of the great circle in Prob. 11.

13. Using the formulae of Prob. 12, Exercise LVIII, find the initial course and distance by great circle from Los Angeles, 34° 03′ N, 118° 15′ W, to Auckland, 41° 18′ S, 174° 51′ E.

14. Find the southern vertex of the great circle in Prob. 13; find also the longitude of its crossing the equator and its direction at that point.

Find the position and course, distance gone and distance to go, of a ship following the track in Prob. 13 when it has

15. covered half the total difference in longitude.

16. covered half the difference in latitude

17. sailed 8.1 days at 15 knots.

18. The sun, in declination $14° 37' S$, is observed by a navigator whose approximate position makes the sun's hour-angle $3^h\ 19^m\ 24^s$ P.M. and his own latitude $10° 13' N$. What are the corresponding altitude and azimuth of the sun? Use the cosine-haversine formula for the altitude, the Sine Law for the azimuth.

19. According to dead reckoning, a navigator is in Lat. $45° S$, and the sun, in declination $22° 30' S$, has the hour-angle 4^h A.M. What are the corresponding values of the sun's altitude and azimuth? Use the formulae developed from Napier's Rules with the perpendicular.

20. If the observer's latitude is $71° 27' N$, and the sun, in declination $21° 18' N$, has the hour-angle $5^h\ 41^m\ 36^s E$, find the corresponding values of the sun's altitude and azimuth, using the cosine-haversine law to compute the altitude, and Napier's Analogies to compute the azimuth. (See Prob. 11, Exercise LXI.)

21. What is the direction of a wall in Lat. $52° 30' N$ which casts no shadow at 6 A.M. (L.A.T.) on the longest day of the year?

22. If a vessel starts directly east from Sandy Hook, Lat. $40° 28' N$, Long. $74° 01' W$, and sails along a great circle at the uniform rate of 16 knots, find its latitude and longitude (1) at the end of 48 hours' sailing; (2) at the end of 5 days' sailing.

23. Find the latitude and longitude of the most northerly point reached by a vessel sailing from San Francisco, Lat. $37° 48' N$, Long. $122° 28' W$, to Calcutta, Lat. $22° 53 N$, Long. $88° 19' E$.

24. In New York, Lat. $40° 43' N$, the sun's altitude is observed to be $30° 40'$, its bearing westerly. The sun's declination is $10° N$. What is the local apparent time?

25. Find the local apparent time of sunrise at Annapolis, 38° 59′ N, when the declination of the sun is (a) 20° N; (b) 20° S.

26. Find the (L.A.T.) time of sunrise at Stockholm, 59° 21′ N, when the declination of the sun is (a) 23° 27′ N; (b) 23° 27′ S. (Longest and shortest days.)

27. Find the (L.A.T.) time of sunrise and sunset at a point 1600 miles from the South Pole when the declination of the sun is 26° S.

28. Find the distance between Regulus and Antares in degrees, their right ascensions and polar distances being, for Regulus 10^h, 77° 19′, and for Antares 16^h 20^m, 116° 06′.

29. At a point where the altitude of the north celestial pole is 38° 59′, the amplitude of a star on the horizon is E 43° 10′ N. Find the polar distance of the star and its hour-angle.

30. On a sunny morning when the declination of the sun was 15° S, it was observed that the house walls along a certain street in Annapolis cast no shadows. What is the direction of the street? The time is 8 A.M. (L.A.T.).

CHAPTER XV

STEREOGRAPHIC PROJECTIONS

135. For many purposes it is desirable to represent the surface of a sphere on a plane. A simple method of doing this is that known as stereographic projection. In this we select a point on the surface of the sphere known as the *center of projection*, and a plane of projection, called the *primitive plane*, through the center of the sphere perpendicular to the diameter through the center of projection. If any other point on the surface of the sphere is joined by a straight line to the center of projection this line intersects the primitive plane in a point which is called the *stereographic projection* of the given point. Occasionally a point not on the surface of the sphere will be projected in the same way; i.e., by drawing a line from it to the center of projection.

The *polar distance of a point* is the angular distance on the surface of the sphere from the pole of the primitive circle opposite the center of projection.

The *polar distance of a circle* is the angular distance on the surface of the sphere of any point in its circumference from its pole.

The *inclination of a circle* is the angle between its plane and the primitive plane.

136. Let $BEDF$ (Fig. 73) be a great circle cut from the sphere by passing a plane through the center O. This plane is the primitive plane and the circle is the primitive circle. The poles of the primitive circle are A and S; in this case the pole S has been chosen as the center of projection.

Let P_1 be any point on the surface of the sphere and through it and the poles of the primitive circle pass a plane. Since this plane passes through the diameter AS it will cut the sphere

in a great circle; in Fig. 73 this great circle is AP_1ES. This plane will intersect the primitive plane in the line EF. Draw P_1S intersecting EF at p_1, the stereographic projection of the point P_1. AP_1 is the true polar distance of the point P_1 and Op_1 is the projected polar distance. Since P_1 was chosen on the surface of the hemisphere which does not contain the center of projection its projection is within the primitive circle. Had P_1 been chosen on the surface between the center of projection and the primitive circle its *projection* would have been *outside the primitive circle.*

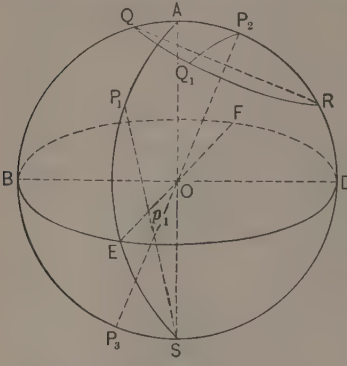


FIG. 73

Let QQ_1R (Fig. 73) be a small circle of the sphere. Through the center of this circle draw the diameter P_2P_3 of the sphere. From Geometry we know that this diameter is perpendicular to the plane of the small circle. The polar distance of this circle is $P_2Q = P_2R = P_2Q_1$.

Let us designate by ϕ the angle which the plane of the circle QQ_1R makes with the primitive plane. Then ϕ is the inclination of QQ_1R and it is measured by the arc AP_2, since the angle between the planes is equal to the angle between the normals OA and OP_2 drawn from O to the two planes. In other words, *the inclination of a small circle is measured by the angle between its pole and the pole of the primitive.*

The line BD, which is the intersection of the primitive plane with a plane passed through the poles of the primitive plane and the pole of the small circle, is defined as *the line of measures for the circle* QQ_1R.

137. Let $ABSD$ (Fig. 74) be a great circle cut from the sphere by a plane through AS and OP, the axes of the primitive

circle and the small circle QR, respectively, and let BD be the straight line in which it intersects the primitive plane. Then BD is the line of measures of the circle QR.

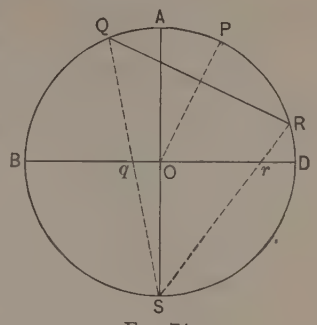


FIG. 74

Since P is the pole of the circle QR, its polar distance is $PQ = PR$ and its inclination is AP. S is the center of projection and q and r are the stereographic projections of Q and R, respectively.

If all the points in the circle QR were connected with the center of projection an oblique circular cone would be formed of which SQ and SR would be the extreme elements.[1]

138. Theorem.—*The stereographic projection of a circle is a circle.*

Let QR (Fig. 75) be any small circle and P the vertex of the cone tangent to the sphere along QR. Choose any point Q on the circle and pass a plane through S, Q, and P. The plane SQP cuts the primitive plane in the line pq which is the projection of the element PQ of the tangent cone. The plane SQP cuts a tangent plane at T

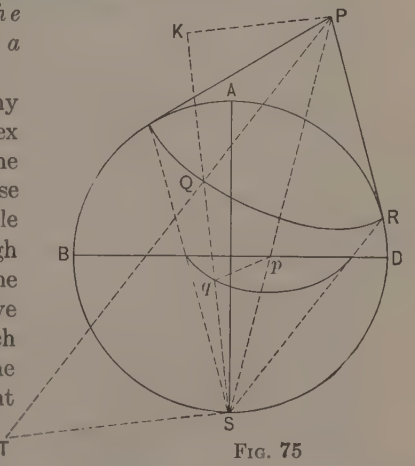


FIG. 75

[1] Extreme elements of a cone are the two elements cut from a cone by a plane through the vertex and the center of the base and which is perpendicular to the base. They are the maximum and minimum elements of the conical surface.

S in the line ST which is parallel to pq; SQP cuts from the sphere a circle which is shown in Fig. 76. Draw PK parallel to pq and ST; continue PQ until it meets TS at T, and draw the line SQ continuing it to meet KP at K.

The alternate interior angles TSQ and QKP are equal.

TS and TQ are tangents to the circle QS from a common point T and are, therefore, equal.

Hence, $\angle TSQ = \angle TQS = \angle KQP = \angle PKQ$.

Therefore, $PK = PQ$.

The triangles Sqp and SKP are similar and we have

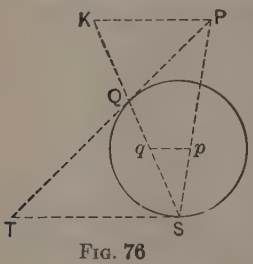


Fig. 76

$$\frac{pq}{PK} = \frac{pq}{PQ} = \frac{pS}{PS}, \text{ or, } pq = PQ \cdot \frac{pS}{PS}.$$

As Q moves around the circumference of the given circle, PQ being an element of a right circular cone is unchanged in length; also P, p, and S being fixed points, the lengths pS and PS are constant.

Therefore, $PQ \cdot \frac{pS}{PS}$ is constant; i.e., pq is a constant.

But p is a fixed point, being the projection of the vertex of the tangent cone.

It follows, therefore, that the locus of q is a circle whose center is the projection of the vertex of the tangent cone.

139. In the exceptional case where the *circle passes through the center of projection, the projection is a straight line.*

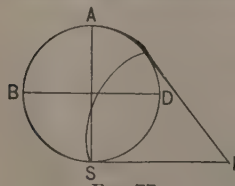


Fig. 77

For, since the circle to be projected passes through the center of projection, all projecting lines drawn to points on the circle lie in the plane of the circle, and hence its projection is the intersection of the plane of these lines with the primitive plane and is a straight line.

Such a case is shown in Fig. 77. Here the element SP is a tangent at S. The projection of the vertex of the tangent cone has been shown to be the center of the projected circle. Here, however, the projecting line is parallel to the primitive. Hence we say the projection of P is at infinity and the radius of the projected circle is infinite, and therefore the projection of the circle is a straight line. This is a special or limiting case of the general theorem that all circles project circles.

In the limiting case where the small circle becomes a great circle, the theorem still holds.

140. Theorem.—*An angle on the surface of a sphere and its stereographic projection are equal.*

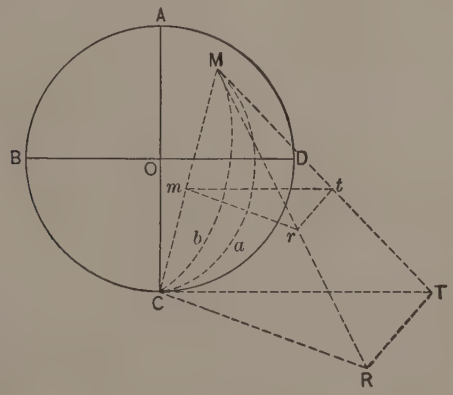


FIG. 78

Let M be the point of intersection of any two curves on the surface of the sphere (these curves are not shown in the figure). At M draw a tangent to each of these curves. These lines will pierce the plane tangent at C in the points T and R respectively. Through T, M, C and R, M, C pass planes. The plane TMC will cut the sphere in the circle MaC, the tangent plane in CT, and will project the tangent line on the primitive plane in the

line mt; similarly RMC will cut the sphere in the circle MbC, the tangent plane in CR, and will project the tangent MR on the primitive in the line mr. Since the tangent plane at C is parallel to the primitive, CR and CT will be parallel to mr and mt, respectively. Draw RT.

Since RC and RM are tangents to the sphere from the same point, RC is equal to RM; similarly, TC equals TM; therefore, the triangles RCT and RMT are equal.

Hence, $\angle RCT = \angle RMT$.

But $\angle RCT$ is equal to $\angle rmt$.

Therefore, $\angle rmt = \angle RMT$.

But the angle between two curves on the surface of a sphere is, by definition, the angle between the tangents drawn at the common point.

That is, *the angle between two lines drawn tangent to a sphere at a common point is equal to the angle between their projections*, or *angles on the surface of the sphere are unaltered in stereographic projection.*

141. The following two basic properties of the stereographic projection have just been proved; viz:

1. That all circles project into circles.
2. That angles are not altered.

These properties enable us to construct the projections of spherical triangles which is the purpose of this discussion.

These principles or properties are fundamental and must be understood thoroughly as all other operations, which are to be introduced, depend upon them.

It is of importance to note that the projection is shown on the side of the primitive plane opposite to that which faces the center of projection; i.e., if the center of projection is at the bottom point of the sphere, as in Fig. 75, the projection is shown on the top side of the primitive plane.

142. In the preliminary projections the following method of lettering will be employed: a point on the surface of the sphere

will be designated by a capital letter, as, P, Q, A, etc.; when a point is projected on the primitive the projected position will be designated by a subscript p as P_p, Q_p, etc; and as it will be necessary to revolve planes in the process of projecting, the revolved position of a point will be designated by a subscript r as P_r, A_r, etc.

143. To Project Any Circle of the Sphere; Given its Polar Distance and the Projection of its Pole.

Let $ABCD$ be the primitive circle, Fig. 79, and E_p the projection of the point E diametrically opposite the center of projection. The center of projection, S, lies in the axis of the primitive directly below E and E_p, and at a distance from E_p equal to the radius of the sphere. Let P_p be the projection of P, the pole of the given circle. Draw the diameter AP_pC and then draw the diameter BD perpendicular to AC. *AP_pC is the projection of a great circle passing through P and the center of projection,* AC is its line of measures.

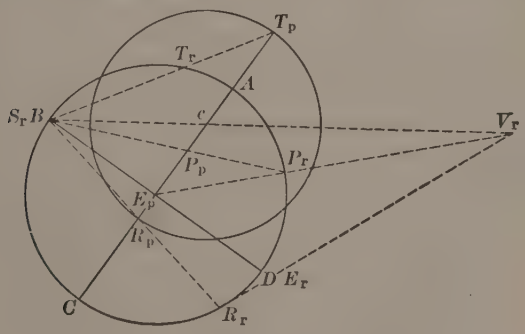


FIG. 79

Revolve the plane of this great circle about the diameter AC into the plane of the primitive. This will bring S, the center of projection, to B; and E, the point opposite the center of projection, to D. P_r the intersection of S_rP_p (produced) with the

primitive circle is the revolved position of P. Lay off from P_r, P_rR_r and P_rT_r each equal to the polar distance of the given circle (since these are lengths along the great circle which has been revolved, they have not been altered). Draw S_rT_r and S_rR_r and they will cut the line of measures in T_p and R_p respectively, i.e., the projected positions of the extremities of a diameter of the given small circle. Then T_p and R_p will be the extremities of a diameter of the projected position of the given circle. Bisect T_pR_p and locate c the center; with c as a center and cT_p as a radius describe a circle. This is the required projection.

The center c can be located in another way. Draw the line E_pP_r and continue it until it meets at V_r a line drawn tangent to the circle at R_r. V_r is the revolved position of the vertex of the tangent cone. Draw S_rV_r which will intersect AC the line of measures at c (hence c is V_p).

The reasoning which justifies this method can best be followed by studying Fig. 75. R_rV_r is the revolved position of an element; E_pP_r is the revolved position of the diameter of the sphere through the center of the given circle; hence the vertex of the tangent cone must lie at the intersection of these lines. But it was shown in Fig. 75 that the line joining the vertex of the tangent cone to the center of projection pierces the line of measures at the center of the projection of the given circle.

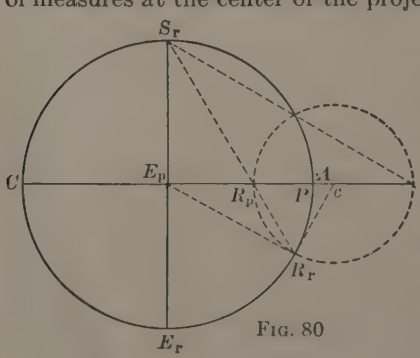


FIG. 80

This offers a second method of constructing the projection of the given circle. That is instead of locating the two extremities of the projected diameter, one extremity and the center can be located, and from these the projected circle can be drawn. This latter

method is particularly convenient when the pole is on the circumference of the primitive circle as in Fig. 80. Here we draw the diameter AC through P and the diameter S_rE_r perpendicular to AC. Here P, P_p and P_r are the same point. From P lay off the polar distance PR_r, draw E_pR_r and the tangent line R_rc meeting the line of measures at c. Then c is the center of the projected circle. Draw S_rR_r meeting the line of measures at R_p. Then with c as a center and a radius cR_p describe the projected circle. In this case c is the vertex of the tangent cone; since it is already in the line of measures its projection must be in coincidence with it.

144. To Project a Great Circle.—When the circle is a great circle its polar distance is 90°. Let P_p be the projection of the pole of a great circle. Through P_p draw the diameter AP_pB and draw the diameter S_rE_r perpendicular to AB. As stated before AB is the projection of a great circle through the center of projection and perpendicular to the axis S_rE_r. Therefore, any great circle with its pole on the great circle of which AB is the projection *must pass through* S_r and E_r. Hence, locate P_r as usual and lay off P_rR_r equal to 90°. Draw the line S_rR_r, and the point R_p where S_rR_r cuts the line of measures, will be one extremity of the diameter of the projection of the great circle. The tangent cone in this case becomes a tangent cylinder of which E_pP_r is the revolved position of the axis. Through S_r draw a line parallel to E_pP_r and designate by c the point where it crosses the line of measures (produced if necessary). This point c is the center of the projected position of the great circle. With c as a center and cR_p as a radius describe the circle.

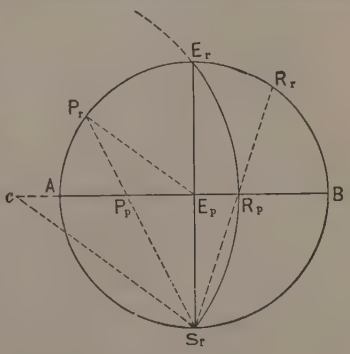


FIG. 81

In thus locating c the same principle was employed as in the case of the small circle. The tangent cone became a cylinder and the line joining the center of projection to the vertex of the tangent cone became in this case a line through the center of projection parallel to an element of the cylinder.

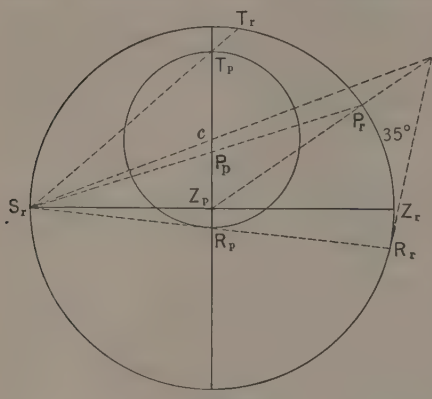


FIG. 82

The center could have been located in other ways. Since S_r, R_p and E_r are all on the circumference of the required projection a line bisecting and perpendicular to either S_rR_p or E_rR_p would have cut the line of centers at c.

EXERCISE LXIII

Taking Z as the point opposite the center of projection, P as the pole of the given circle, and ϕ as the angle between the plane of the given circle and the primitive, draw the following circles.

(1) $PZ = 35°$, p (polar distance) $= 47°$. (See Fig. 82.)

Here $P_rZ_r = PZ = 35°$; draw S_rP_r thus locating P_p. From P_r lay off $P_rR_r = P_rT_r = 47°$; locate T_p and R_p. Bisect T_pR_p and locate c; or draw tangent line at R_r meeting Z_pP_r produced at V_r and draw S_rV_r thus locating c.

(2) $PZ = 40°$, $p = 90°$. (See Fig. 83.)

Lay off $Z_rP_r = PZ = 40°$ and locate P_r; lay off $P_rR_r = 90°$ and locate R_p. Through S_r draw a line making an angle of 40° with S_rZ_r (this can easily be done by making $LZ_r = 80°$) and locate c; with c as a center and radius cR_p draw the circle. Since this is a great circle it should pass through S_r and Z_r.

The portion of the projected circle which is outside of the primitive is the projection of that portion of the circle which lies between the center of projection and the primitive plane; the portion inside the primitive is the projection of that portion of the circle which is on the opposite side of the primitive.

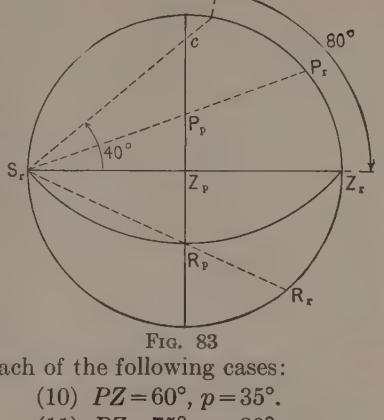


FIG. 83

1. Project the circle in each of the following cases:

(1) $p=30°$, $\phi=45°$.
(2) $p=45°$, $\phi=60°$.
(3) $p=30°$, $\phi=15°$.
(4) $p=60°$, $\phi=30°$.
(5) $p=45°$, $\phi=45°$.
(6) $p=90°$, $\phi=30°$.
(7) $p=60°$, $\phi=0°$.
(8) $PZ=55°$, $p=45°$.
(9) $PZ=35°$, $p=60°$.
(10) $PZ=60°$, $p=35°$.
(11) $PZ=75°$, $p=30°$.
(12) $PZ=90°$, $p=45°$.
(13) $PZ=90°$, $p=30°$.
(14) $PZ=30°$, $p=90°$.
(15) $PZ=50°$, $p=90°$.
(16) $PZ=90°$, $p=90°$.
(17) $PZ=0°$, $p=90°$.
(18) $PZ=20°$, $p=90°$.

2. Show where the proof that circles project into circles breaks down when the given circle is a great circle.

145. To find the Locus of the Centers of the Projections of All Great Circles Passing through a Given Point.

Let P_p (Fig. 84) be the projection of the given point through which the projections of great circles are to pass; draw the diameters AP_pC and S_rE_r at right angles. Draw $S_rP_pP_r$.

(1) The projections of all great circles through P_p must also pass through the projection of the point Q, 180° from P. Hence draw the diameter P_rQ_r and project Q_r to the primitive by drawing the line S_rQ_r which cuts the line of measures AC in Q_p.

Since all the required projected circles pass through P_p and Q_p the locus of their centers must be the perpendicular bisector

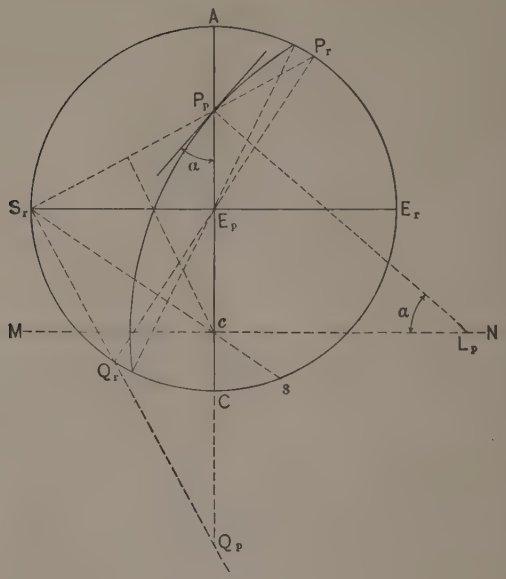


FIG. 84

of the line P_pQ_p. The line MN in Fig. 84, drawn perpendicular to P_pQ_p at the point c is the required line. It is called the line of centers for the point P.

(2) The triangle S_rcP_p is isosceles and the angle P_pS_rs equals the angle S_rP_pC which is measured by $\frac{1}{2}(90°+P_rA)$ $=\frac{1}{2}P_rAS_r$: i.e., the arc$P_rE_rs=P_rAS_r$. Hence lay off P_rE_rs $=P_rAS_r$ and draw S_rcs. This is equivalent to laying off a polar distance P_rAS_r from P_r; and thus the line of centers is the projection of a small circle passing through the center of projection and having a polar distance $P_rAS_r=180°-\phi$, where $\phi = E_rP_r$ denotes its inclination.

(3) From the construction $S_rQ_r=P_rE_r$ and $Q_rCs=180°-$ $P_rE_rs=180°-P_rAS_r=P_rE_r$. Hence, lay off $S_rCs=2P_rE_r$

and draw S_pcs. This method usually gives the most accurate results.

146. To Draw the Projection of a Great Circle through a Point P making a given Angle with the Great Circle through P and the Center of Projection.

In Fig. 84 let P_p be the projection of the given point; AC will be the projection of the great circle through P and S; let us denote the given angle by α. Draw the line of centers as just described. A tangent at P to the great circle of which AC is the projection would project into AC; a tangent to the required circle at P would project into a straight line in the primitive passing through P_p. Since angles are unaltered in the stereographic projection the angle between these two projected lines would be α.

Hence if we draw a perpendicular to the tangent at P_p it falls along the radius of the required projection. This perpendicular meets the line of centers at L_p. Hence L_p is the center of the projection of the circle and L_pP_p is the radius.

In constructing the figure it is not necessary to draw the tangent line; merely lay off angle $E_pP_pL_p$ equal to $90° - \alpha$.

It is sometimes convenient to solve this problem by the principle that the angle between two great circles is equal to the angular distance between their poles.

A special case is one where the point is on the primitive circle. Given the point P as in Fig. 85 to draw the projection of a great circle making an angle α with the primitive circle.

A great circle through P will also pass through Q, a point 180° from P. From P measure an angle 2α on the primitive circle and locate A. Draw AQ; where AQ intersects DE_r will be the center of the required circle; with c as a center and cP as a radius draw the projected circle. For the construction, it is well to note that BQ bisects the angle AQP.

The proof that c is the required center follows: DE_r is the line of centers for the point P; hence the required center lies

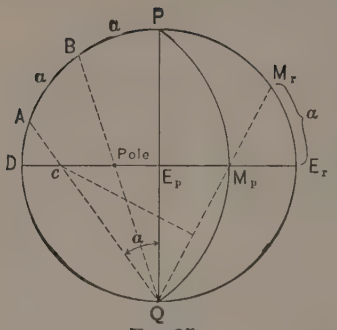


FIG. 85

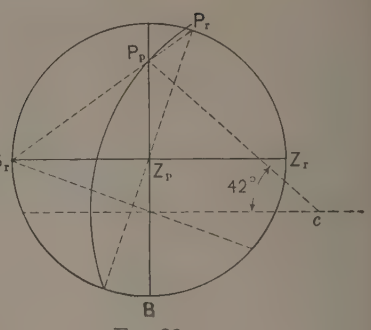


FIG. 86

somewhere on this line: $\angle AQP$ is measured by $\frac{1}{2}$ arc $AP = a$. Since the desired great circle makes an angle a with the primitive it makes an angle $90° - a$ with PEQ. Tangents to these latter circles would make an angle $90° - a$: therefore their radii would make the same angle, hence $E_p cQ$ must be equal to $90° - a$, or AQP must equal a.

Sometimes it is more convenient to make this construction by laying off arc $E_r M_r = a$; draw QM_r locating M_p; draw the perpendicular bisector of QM_p and this latter line will cross DE_r at c. The proof of this is left as an exercise for the student.

EXERCISE LXIV

1. Given $PZ = 70°$ (P is the pole of the circle, Z is the point opposite the center of projection) draw the projection of a great circle making an angle of 42° with PZ. (See Fig. 86.)

Measure $Z_r P_r = PZ = 70°$ and locate P_p; find line of centers for P_p. Measure angle $Z_p P_p c = 90° - 42°$ thus making $\angle c = 42°$. With c as center and cP_p as radius describe the circle. Check line of centers by drawing a line through the two points where the projected circle cuts the primitive; this line should pass through the center of the primitive.

2. $PZ = 70°$; draw a great circle through P making an angle of 30° with PZ.

3. $PZ = 50°$; find by three methods the line of centers for the point P.

4. $PZ = 40°$; draw a great circle through P making 75° with PZ.

Draw line of centers of P for the following:

5. $PZ = 70°$.

6. $PZ = 45°$.

7. $PZ = 80°$.

8. $PZ = 30°$.

9. $PZ = 90°$.

10. $PZ = 70°$; draw a great circle through P making an angle of 50° with PZ.

11. $PZ = 50°$; draw a great circle through P making an angle of 50° with PZ.

12. $PZ = 90°$ (i.e., P on the primitive); draw a great circle through P making an angle of 20° with the primitive.

13. $PZ = 90°$; draw a great circle through P making an angle of 65° with the primitive.

147. To Find the Pole of a Given Circle.

(*a*) *A Great Circle.* (Fig. 81). Let $S_rR_pE_r$ be the projected position of the given great circle. Draw S_rE_r and at right angles to it draw the diameter AB. Since the projected great circle cuts the line of measures at R_p, this point must be the projection of the extremity R of a diameter of the given circle. Hence R_r must be a revolved position of R. Since the circle is a great circle the pole must be 90° from R; therefore, measure R_rP_r equal to 90° and P_r will be the revolved position of the pole. Draw S_rP_r and P_p, where S_rP_r cuts AB the line of measures, will be the projected position of the pole of the given great circle.

(*b*) *A Small Circle.* (Fig. 79).

Let c be the center of the projected position T_pR_p of the given small circle; through c draw the diameter AC and draw S_rE_r

the perpendicular to it. Draw S_rR_p and S_rT_p intersecting the circumference of the primitive at R_r and T_r respectively. Bisect the arc T_rAR_r in P_r and draw S_rP_r intersecting the line of measures AC at P_p, the required pole.

148. There is a theorem of Solid Geometry which is of importance at this point; namely, *if a plane passes through the poles of two great circles it cuts off equal arcs on the two circles.* We shall proceed to prove this.

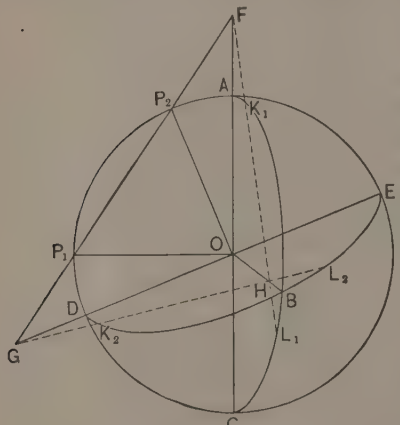


FIG. 87

In Fig. 87 let P_1 and P_2 be the poles of the great circles ABC and DBE respectively. O is the center of the sphere; draw OP_1, OP_2 and P_1P_2 and continue P_1P_2 to meet AC (produced) at F and DE (produced) at G.

Since $OP_1 = OP_2$ the triangle OP_1P_2 is isosceles.

Therefore,

$$\angle OP_1P_2 = \angle OP_2P_1.$$

Hence the right triangles GOP_2 and FOP_1 are equal and $OF = OG$.

Pass any plane as GFH through P_1 and P_2, the poles of the given circles, cutting OB at H. Draw FK_1HL_1 and GK_2HL_2. Then the right triangles FOH and GOH are equal and $\angle HFO$ equals $\angle HGO$.

The arc AK_1 will, therefore, equal the arc DK_2 and the arc CL_1 will equal the arc EL_2 since F and G are the same distance from their respective great circles.

But the arc $K_2BL_2 = 180° - (DK_2 + EL_2)$ and the arc $K_1BL_1 = 180° - (AK_1 + CL_1)$. Therefore, the arc K_2BL_2 is equal to the arc K_1BL_1.

149. Given the Projected Arc of a Great Circle, to Find its True Length.

Let AM_p be the projected arc to be measured. Find the pole P_p of the great circle $AM_pR_pS_r$ as previously described. Draw the line P_pM_p meeting the primitive circle at the point M_r the revolved position of M. Then AM_r is the required length. For $P_pM_pM_r$ is the projection of a small circle through the center of projection S. This small circle passes through the poles of two great circles (i.e., S the pole of the primitive and P the pole of $AMRS$) and by the proposition of Art. 148 it intercepts equal arcs on these circles. Therefore, $AM_p = AM_r$; but AM_r is measured on the primitive circle, and hence is a true length.

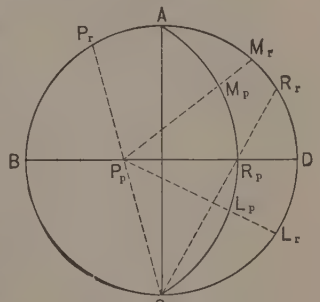


FIG. 88

In a similar manner M_rL_r is the true length of M_pL_p; M_rD is the true length of M_pR_p, etc.

Hence, *to measure the true length of an arc of a projected circle, locate the projected pole of the circle and lines drawn through the projected pole and the extremities of the projected arc will intercept the true length of that arc on the primitive circle.*

The solution of the reverse problem is simply to lay off the true length on the primitive circle beginning at the correct point and draw lines to the pole. These lines will intercept on the projected great circle the projection of the given length. For example to lay off from M the length ML; measure $M_rL_r = ML$ from M_r and locate L_r; draw P_pM_r and P_pL_r, and M_pL_p will be the required projected length.

EXERCISE LXV

1. $PZ = 50°$. A great circle through P makes an angle of 30° with the great circle through PZ and cuts the primitive plane

at A and B. Find the true length of AP and PB. (Z is the point opposite the center of projection.)

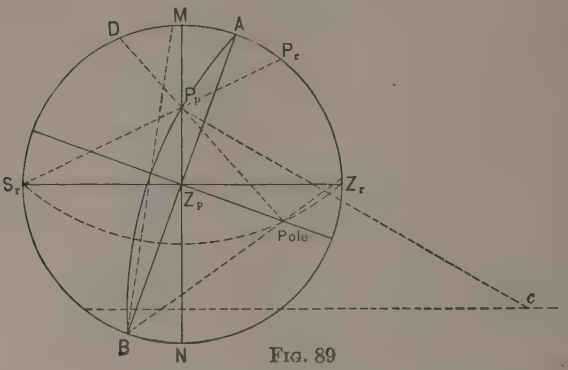


Fig. 89

Lay off $Z_rP_r = ZP = 50°$; draw S_rP_r and locate P_p. Find line of centers of P_p. Lay off angle $NP_pc = 60°$ and locate c. Draw arc AP_pB using c as a center and cP_p as a radius. Draw diameter AB and locate pole of AP_pB. Draw a line from this pole through P_p and let D be the point where it crosses the primitive. AD and BD are the required lengths.

$AD = 43°30'$, $BD = 136°\ 30'$. These results were measured on the figure.

2. The projection of a great circle through S_r and Z_r cuts the diameter perpendicular to S_rZ_r at A_p, 25° from one end. Mark B_p the mid-point of the projection A_pZ_r and find the true lengths of the arcs AB and BZ_r.

3. $PZ = 55°$. A great circle through P makes an angle of 70° with PZ and cuts the primitive plane at A and B. Find the true lengths AP and PB.

4. Same as **3** except $PZ = 30°$, circle makes 60° with PZ.

5. $PZ = 35°$. Draw through P a great circle making 45° with PZ and cutting S_rZ_r at A_p, and primitive at B and C. Measure PA, AB, and PC.

150. When any side and two other parts of a plane triangle are known, we are able to construct the triangle with the aid of the rule, compasses and protractor. In plane triangles the construction involves the following problems:—

(1) From a given point on a given line to lay off a given length along the line; or to draw a circle, given the center and radius.

(2) Through a given point on a given line to draw a second line which shall make a given angle with the first.

In the stereographic projection the analogy to this is very close and these problems become:

(1a) To lay off from a given point on a given great circle a given arc along the circle; or to draw a circle given its pole and polar distance.

(2a) Through a given point on a great circle to draw a second great circle which shall make a given angle with the first.

151.[1] **To Lay Off a Given Arc Along a Given Great Circle.** In Fig. (90) let PML be the given great circle and Qr the given arc; describe a circle about P as a pole, with the given

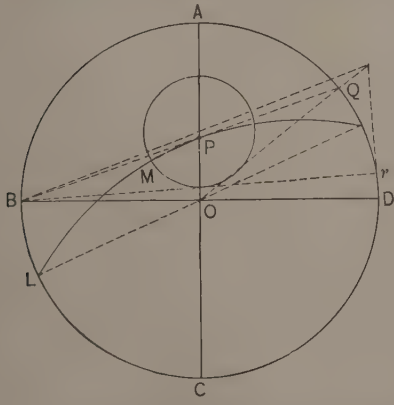


Fig. 90

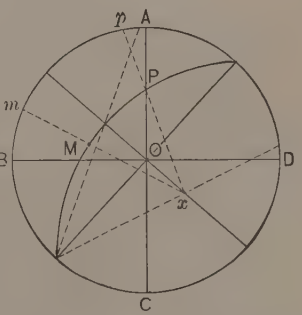


Fig. 91

[1] From this point the subscript notation will no longer be used.

polar distance Qr; the circle will cut PML in M, making $PM=Qr$.

Or we may proceed as in Fig. 91: Find the pole x of the given circle and draw xPp; lay off $pm=Qr$, and draw xMm; then $PM=Qr$.

To Construct a Given Angle.

Let PM, Fig. 92, be the given great circle; it is required to draw through P a great circle which shall make a given angle with PM. Find the line of centers cmK of the point P and draw the radius mP; through P draw the line PK making the given angle with Pm, and meeting cm at K; then K is the center and KP the radius of the required circle. For the angle between the circles is measured by the angle between their tangents at the common point, and hence by the angle between their radii at that point.

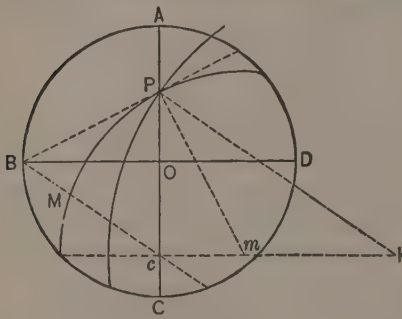


Fig. 92

152. We have now demonstrated all of the preliminary operations essential to the projection of triangles, which is the purpose of this study. Before actually projecting a triangle it is recommended that a small triangle be drawn giving the essential information for the projection, as will be shown in the following illustration.

(1) Given two sides and the included angle; or b, c and A. In Fig. 93 the triangle is projected with A at the center; in Fig. 94 a side is placed in the circumference. In this illustration $b=40°$, $c=60°$, and $A=65°$.

Construction.—Draw small triangles giving dimensions. At A lay off the angle $A=65°$, lay off $AB=c=60°$, and $AC=b$

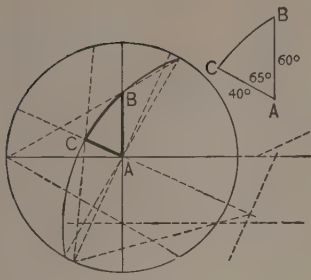


FIG. 93

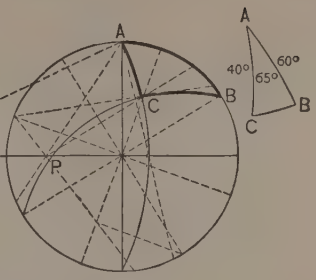


FIG. 94

$=40°$; find lines of centers for B and C respectively; with the point of intersection of these two as a center, draw BC.

To measure unknown parts.—With B as a pole describe a great circle; arcs BC and BA (produced if necessary) will both intersect this great circle; from B project these two points to the primitive and measure the distance between these two projected points; this will measure $\angle B$. Similarly C can be measured. Find the pole of BC and project B and C to primitive from this point; this will give the true length of a.

(2) Given two sides and the angle opposite one of them or a, b, and A. In Fig. 95 the triangle is projected with A at the center; in Fig. 96 a side is in the circumference. In this illustration $a=35°$, $b=60°$, and $A=30°$

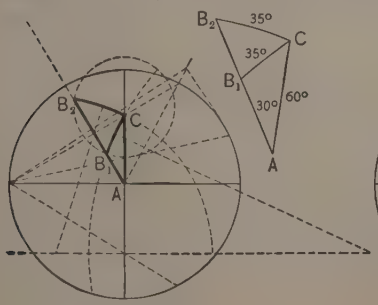


FIG. 95

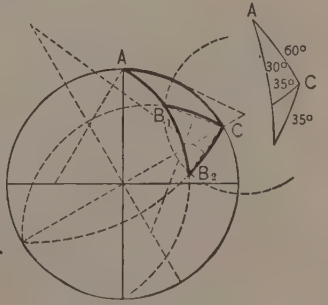


FIG. 96

Construction.—Draw small triangles giving dimensions. At A lay off angle $A = 30°$, $AC = b = 60°$; with C as a pole and polar distance $a = 35°$ describe an arc cutting AB in B_1 and B_2; complete the triangles by drawing the arcs CB_1 and CB_2. In this case the line of centers for C was used and the centers were located by drawing the perpendicular bisectors of the arcs B_1C and B_2C.

There will be *two* solutions when the circle about C cuts AB twice on the same side of A as is C; *one* solution when it cuts AB only once on that side; and *no* solution when the circle does not cut AB.

The unknown parts can be measured as described in Ex. (1).

(3) Given the three sides; or a, b and c. In Fig. 97 the triangle is projected with A at the center; in Fig. 98 a side is placed in the circumference of the primitive. In this illustration $a = 45°$, $b = 75°$ and $c = 60°$.

Lay off the side b from A to C; then with A and C as poles and with polar distances respectively equal to c and a construct circles meeting at B. The sides AB and BC are then drawn by passing great circles through A and B, and B and C.

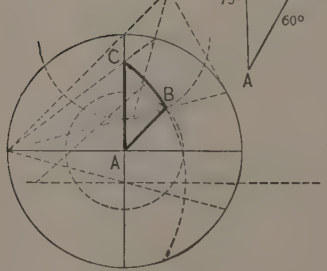


FIG. 97

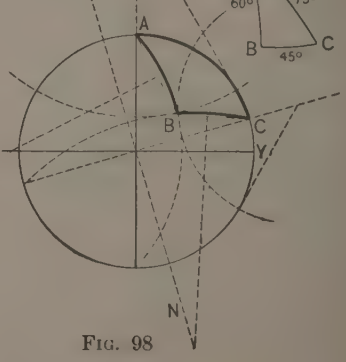


FIG. 98

To measure the three angles draw the great circles of which A, B, and C are the poles. For A, Fig. 97, this will be the primitive; continue AB and AC to the primitive and measure the intercepted arc: for C, draw the great circle of which C is the pole and from C project to the primitive circle the points where AC and CB intersect this arc; similarly, for B, draw the great circle of which it is the pole and project to the primitive the points where BC and BA cut this arc. For A, Fig. 98, the great circle of which it is the pole is the line XY, hence project to the primitive the points where AB and AC cut this line; for C, MN is the polar great circle, hence project to the primitive circle the points where AC and CB cut this line; for B, draw the great circle of which B is the pole and project to the primitive circle the points where AB and BC cut this arc.

Other cases can be worked out in a similar manner although these are all that are really necessary, as by means of polar triangles the others may be reduced to these.

EXERCISE LXVI

Construct the following triangles and measure the unknown parts, given:—

1. $A=60°$ $B=45°$, $c=60°$.
2. $b=45°$, $c=60°$, $A=60°$.
3. $a=30°$, $b=60°$, $A=30°$.
4. $A=105°$, $B=60°$, $b=52°\ 30'$.
5. $A=120°$ $B=45°$, $C=60°$.
6. $A=95°\ 30'$, $b=64°\ 30'$, $a=100°$.
7. $A=116°$, $B=60°$, $c=52°\ 30'$.
8. $a=56°\ 30'$, $b=104°\ 15'$, $c=75°\ 30'$.
9. $c=65°\ 30'$, $b=112°\ 30'$, $A=45°\ 21'$.

CHAPTER XVI

STEREOGRAPHIC PROJECTIONS OF THE TERRESTRIAL AND CELESTIAL TRIANGLES

153. The applications of the stereographic projections will be to the terrestrial and celestial spheres. In either case the sphere will be considered to have shrunk to a sphere whose diameter is equal to that chosen for the primitive circle.

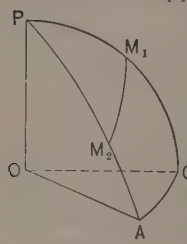


FIG. 99

154. One of the important applications is to the "Course and Distance" problem; i.e., to find the course and shortest distance on the earth's surface between two places whose latitudes and longitudes are given (see Art. 128).

Let P be the pole, O the center of the earth, M_1 and M_2 the two places and AC the equator; PC is the meridian of M_1 and PA the meridian of M_2. The angle between the two meridians is the algebraic difference of the longitudes of the two places, if this is less than 180°. If it is greater, M_1PM_2 is 360° minus this difference; CM_1 is the latitude and PM_1 the co-latitude of M_1; AM_2 is the latitude and PM_2 the co-latitude of M_2.

Hence to project the triangle PM_1M_2 is an application of the case of two sides and the included angle. Ordinarily it is simplest to use the meridian of the point of departure as the primitive circle. However, if the algebraic difference of the longitudes is more than 45° and less than 135° the center of the projection of the meridian of the point of destination is outside the primitive circle. The nearer this angle is to 90° the further away is the center. In such a case project on a meridian other than that of departure, the equator, or on the great circle of which the point of departure is the pole.

Illustration.—Find the great circle course and distance from Vancouver (49° 16′ *N*, 123° 11′ *W*) to Yokohama (35° 27′ *N*, 139° 40′ *E*). Here the difference of longitudes is 360° − (139° 40′ + 123° 11′) = 97° 09′. The projection is made on a great circle 45° *E* of Vancouver.

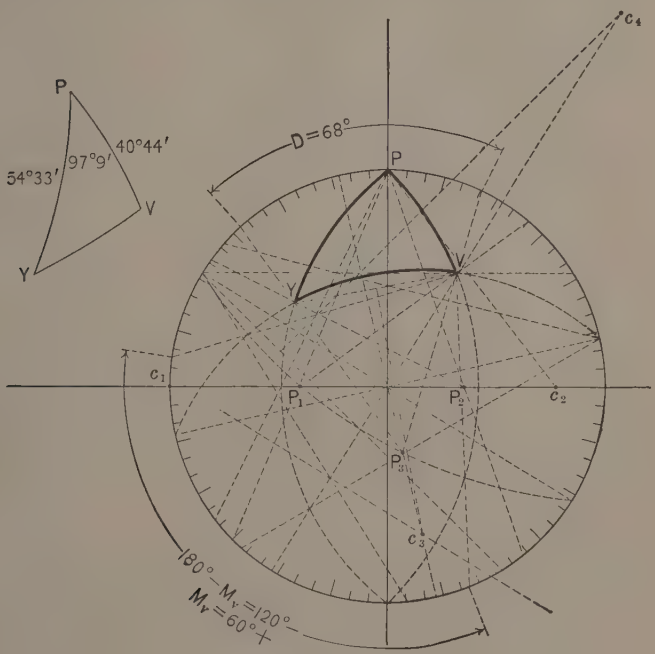


FIG. 100

EXERCISE LXVII

Find the great circle course and distance from

1. Gibraltar (36° *N*, 06° *W*) to Cape Henry (38° *N*, 76° *W*).

2. Cape Flattery (48° *N*, 8 hrs. 20 m. *W*) to Java (9° *S*, 7 hrs. 28 m. *E*).

3. San Francisco (37° 50′ N, 8 hrs. 10 m. W) to Manila (14° 36′ N, 8 hrs. 4m. E).

4. New York (40° 40′ N, 4 hrs. 55 m. 54 s. W) to Cape of Good Hope (33° 56′ S. 1 hrs. 13 m. 55s. E).

5. San Francisco (37° 50′ N, 8 hrs. 10 m. W) to Sydney (33° 52′ S. 10 hrs. 04 m. 50s. E).

(In the following examples L means latitude; λ, longitude.)

6. $A(L=15°\ S,\ \lambda=3\text{h}\ W)$ to B $(L=60°\ N,\ \lambda=11\frac{1}{2}\ \text{h}.\ W)$.

7. $A\ (L=0°,\ \lambda=3\ \text{h}\ E)$ to $B(L=20°\ S,\ \lambda=6\ \text{h}\ W)$.

8. $A\ (L=40°\ N,\ \lambda=4\frac{1}{2}\ \text{h}\ E)$ to $B\ (L=60°\ S,\ \lambda=3\ \text{h}\ W)$.

9. $A\ (L=30°\ S,\ \lambda=2\ \text{h}\ E)$ to $B\ (L=55°\ N,\ \lambda=7\frac{1}{2}\ \text{h}\ W)$.

10 $A\ (L=20°\ N,\ \lambda=1\ \text{h}\ W)$ to $B\ (L=30°\ N,\ \lambda=4\frac{1}{2}\ \text{h}\ W)$.

11. $A\ (L=10°\ N,\ \lambda=1\ \text{h}\ E)$ to $B\ (L=45°\ N,\ \lambda=3\ \text{h}\ W)$.

12. $A\ (L=15°\ N,\ \lambda=3\ \text{h}\ W)$ to $B\ (L=40°\ N,\ \lambda=5\frac{1}{2}\ \text{h}\ W)$.

13. $A\ (L=50°\ N,\ \lambda=2\ \text{h}\ E)$ to $B\ (L=10°\ S,\ \lambda=2\ \text{h}\ W)$.

14. $A\ (L=17°\ N,\ \lambda=100°\ W)$ to $B(L=38°\,05'\ S,\ \lambda=150°\ E)$.

15. A ship receives her radio bearings from Bar Harbor Station 109°, and from Cape Hatteras Station 66°; project on the meridian of Cape Hatteras Station and find the position of the ship. The latitude and longitude of Bar Harbor Station are respectively 44° 19′ N and 68° 11′ W, of Cape Hatteras Station 35° 14′ N and 75° 32′ W.

155. The Astronomical Triangle.—The spherical triangle of which the three vertices are respectively P the pole, Z the zenith and M any point on the celestial sphere has been defined as the astronomical triangle. The astronomical triangle can be projected on the horizon, meridian, prime vertical or the equator. In any case a sketch of the figure should be made as in Fig. 101 and the dimensions supplied before the projection is made. The problem thus reduces to one of the geometrical projections previously demonstrated.

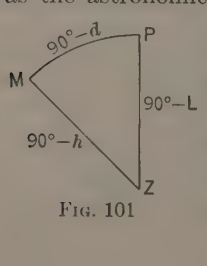


FIG. 101

156. Case 1: Given Latitude, Declination, and Altitude.
Given $L=30°$ S, $d=15°N$, $h=30°$ bearing W.

The projection in the illustration is on the meridian. This is an example with three sides given, one of which is in the primitive.

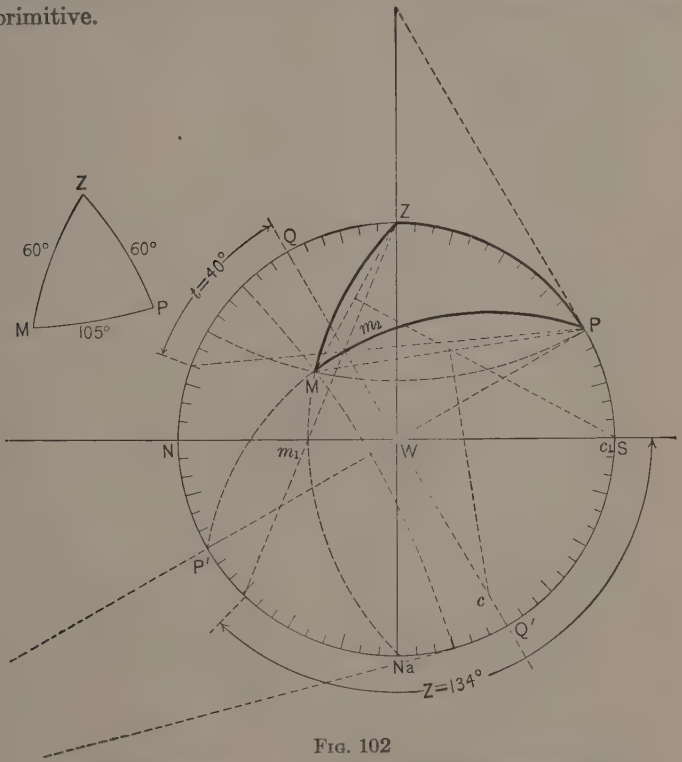


FIG. 102

$PQ\ P'Q$ = Meridian.
Mm_1 = Altitude of M.
$Q'WQ$ = Equator.
PMP' = Hour Circle of M.
Mm_2 = Declination of M.
$MPZ = t$ = Hour angle of M.
NWS = Horizon.
ZMm_1 = Vertical Circle of M.
PZM = Azimuth of M.

Draw the rough sketch and then project.

With Z as a pole a circle is drawn with $p = 60°$; since declination of M is north a circle is drawn with P' as a pole, of polar distance 75°; the intersection of these two is M. The center of $ZMNa$ is c_1; the center of $P'MP$ is c. Z is the pole of NS, hence to measure Z, project m_1 and S to primitive from Z as a pole; P is the pole of QQ', hence to measure t project m_2 and Q to primitive from P as a pole.

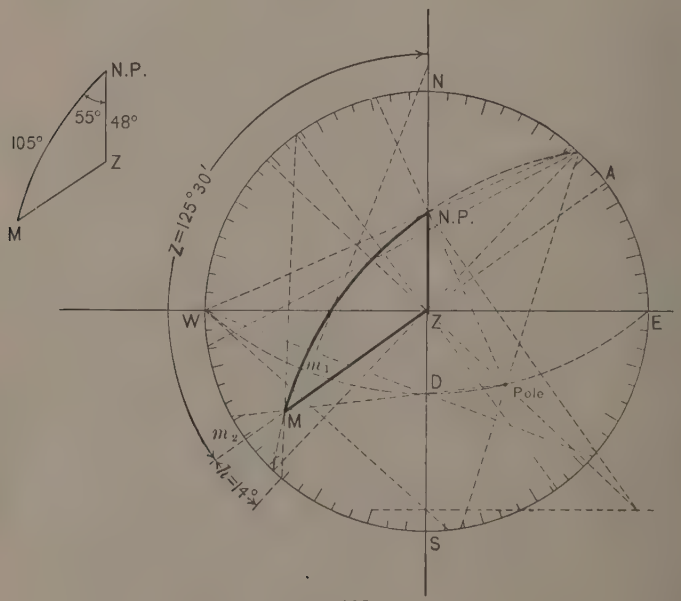


FIG. 103

NZS = Meridian.
Mm_2 = Altitude of M.
WDE = Equator.
PM = Hour Circle of M.
Mm_1 = Declination of M.
MPZ = Hour angle of M.
$NESW$ = Horizon.
ZMm_2 = Vertical Circle of M.
PZM = Azimuth of M.

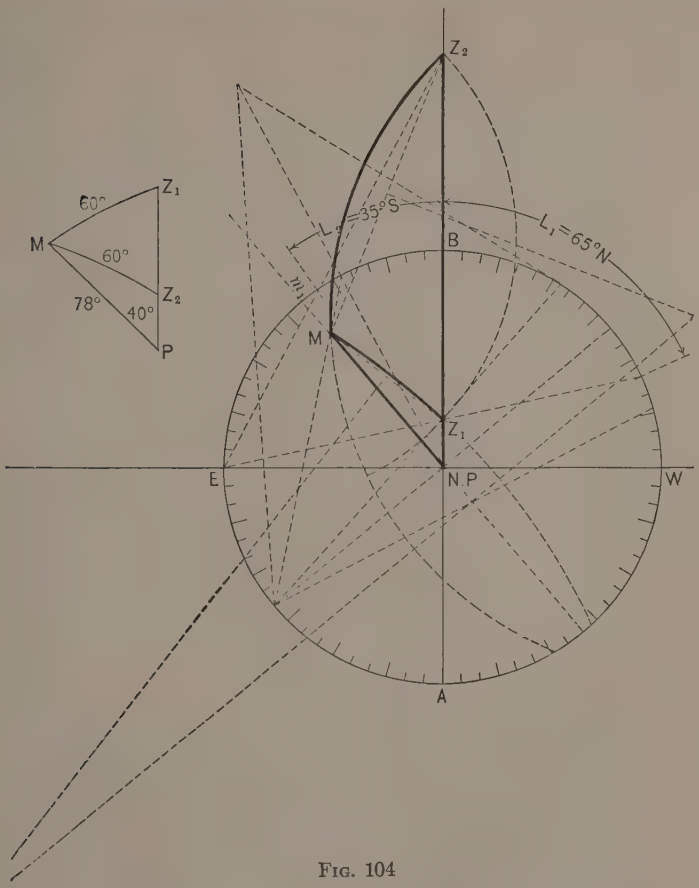


FIG. 104

$L_1 = 65°$ N.
$L_2 = 35°$ S.
Z_2A = Meridian.
BZ = Latitude of M.
$BWAE$ = Equator.
$N.P.-M$ = Hour Circle of M.
Mm_1 = Declination of M.
MPZ = Hour Angle of M.
ZM = Vertical Circle of M.
PZM = Azimuth of M.

157. Case II: Given Hour Angle, Declination, and Latitude.

Given $L = 42°\ N$, $d = 15°\ S$, $t = 3$ hrs. 40 m., body bearing W. The projection in the illustration is on the Horizon. 3 hrs. 40 m. $= 55°$. (See Fig. 103.)

This is an illustration of the case of two sides and the included angle. Results $h = 14°$, $Z = N\ 125°\ 30'\ W$.

158. Case III: Given Hour Angle, Declination, and Altitude. The purpose in this problem is usually to find the Latitude although the Azimuth can also be measured from the projection. On account of the relative simplicity of the construction this projection is usually made upon the Equator. Given $t = 40°$, $d = 12°\ N$, $h = 30°$, body bearing E, to find L.

This is an illustration of the case of two sides and the angle opposite one of them. (See Fig. **104**.)

It will be noticed that Z_2 falls outside the primitive circle. Since the center of the primitive is the North Pole this means that the latitude is south. The Azimuth of M could be found by drawing the great circle of which Z is the pole, i.e., the horizon. As there are two possible zeniths there would be two values for the Azimuth, one at Z_1 and the other at Z_2.

EXERCISE LXVIII

1. Given $L = 40°\ N$, $d = 20°\ N$, $h = 22\frac{1}{2}°\ W$; project the triangle on the plane of the horizon and measure t and Z.

2. Given $L = 30°\ N$, $d = 15°\ S$, $t = 60°$ bearing E; project on the plane of the equator and measure Z and h.

3. Given $t = 40°$, $d = 12°\ N$, $h = 30°$, body bearing East; project on the equator and find L.

4. Given $L = 30°\ N$, $d = 15°\ N$, $t = 45°$ bearing W; project on the meridian and measure h and Z.

5. Given $L = 30°\ N$, $d = 15°\ N$, $h = 45°\ W$; project on the plane of the horizon and measure t and Z.

6. Given $t=30°$, $d=25°$ N, $h=45°$, body bearing West; project on the equator and find L.

7. Given $L=45°$ N, $d=15°$ N, $t=37°$ $30'$ bearing W; project on the plane of the Meridian and measure Z and h.

8. Given $L=40°$ N, $h=0°$, $d=20°$ N; project on horizon and measure the azimuth and hour angle (time of sunset).

9. Make a projection of the Astronomical Triangle on the Meridian (west side) and indicate clearly the latitude of the observer, the equator, the horizon, zenith, nadir, meridian, prime vertical, vernal equinox, a vertical circle, and the star's declination, right ascension altitude, zenith distance, polar distance, hour angle and Azimuth.

10. Given $L=25°$ N, $d=8°$ S, $h=40°$ W; project on the equator and measure t and Z.

11. Given $t=30°$, $h=60°$, $d=15°$ N, body bearing West; project on the Equator and measure L.

12. Given $L=25°$ N, $d=52°$ N, $t=38°$, body bearing W; project on the plane of the meridian and measure Z and h.

13. Given $L=35°$ N, $d=10°$ S, $h=25°$, body bearing W; project on the plane of the horizon and measure t and Z.

14. Given $t=32°$ W, $d=21°$ N, $h=40°$; project on the plane of the meridian and measure L and Z.

Hint:—Locate P first, then M, and with M as a pole describe an arc of radius co-h thus locating Z.

15. Given $t=30°$ W, $d=20°$ N, $h=45°$; project on the plane of the equator and measure L.

16. Given $t=30°$ E, $d=22\frac{1}{2}°$ N, $h=15°$; project on the plane of the equator and measure L.

17. Given $t=60°$ E, $d=15°$ N, $h=20°$; project on the plane of the meridian and measure L.

18. Given $t=60°$ W, $d=30°$ S, $h=18°$; project on the plane of the equator and measure L.

19. Given $t=60°$ W, $d=30°$ N, $h=15°$; project on the plane of the meridian and measure L.

20. Given $t=30°$ E, $d=25°$ N, $h=45°$; project on the plane of the meridian and measure L and Z.

21. Make a projection on the horizon for Problem 10 of Ex. LIX.

For further exercises make projections of the problems in Exercises LVII, LIX, LX, and LXI.

ANSWERS

BULLARD AND KIERNAN TRIGONOMETRY

(REVISED)

Exercise I

1. 3, 4, 5, $\frac{5}{2}$.
2. $\frac{1}{2}$, $\frac{1}{3}$, -3, -2.
3. 1000, 9, 0.0001.
4. 3, 27, $\sqrt{12}$.
5. 1, 1, 1, 0, 0.
11. (*a*) $+\frac{1}{\sqrt{2}}$; (*b*) $\frac{1}{9}$.

Exercise II

1. 1.47712.
2. 1.99123.
3. 0.15490.
4. 0.51931.
5. 2.72016.
6. 0.09542.
7. 0.36179.
8. 0.86180.
9. 0.25105.
10. 0.91321.
11. 0.51877.
12. 8.11625.
13. 2.40824.
14. 2.24165.
15. 0.11179.
16. 0.37265.

Exercise III

1. 1.
2. 4.
3. 0.
4. 7 − 10.
5. 0.
6. 9 − 10.
7. 5.
8. 9 − 20.
9. 0.
10. 8 − 10.
11. 2.
12. 6 − 10.
13. 5.
14. 7 − 10.

Exercise IV

7. 3.53479.
8. 1.85126.
9. 0.90309.
10. 0.89509.
11. 0.49715.
12. 1.48391.
13. 7.87696 − 10.
14. 1.03411.
15. 2.43430.
16. 1.00000.
17. 2.00864.
18. 9.56126 − 10.
19. 3.87545.
20. 5.95428.
21. 1.08001.
22. 9.08647 − 10.
23. 7.33726 − 10.

Exercise V

1. 314.42.
2. 66.891.
3. 1,443,200.
4. 212,800,000.
5. 9513.2.
6. 0.0012377.
7. 1587.8.
8. 2.0894.
9. 200,000.
10. 0.30000.
11. 314.16.
12. 0.00010163.

Exercise VI

1. 0.19835.
2. 2240.9.
3. 31.236.
4. 31.311.
5. 0.099452.
6. −34.536.
7. 1.3152.
8. 1.0383.
9. 15.980.
10. 56.369 sq. ft.

Exercise VII

1. 0.0027360.
2. −0.00013996.
3. −4.4814.
4. −2.2222.
5. −0.0041152.
6. 0.00000064296.
7. 4.6987.
8. −0.033992.
9. −2,712,100.
10. −0.13030.
11. 0.41021.
12. 0.41374.
13. 949.14.
14. 1,025.8.

Exercise VIII

1. 1.5373.
2. 1.7896.
3. 2.3666.
4. 2.7183.
5. 6.0110.
6. 6.5764.
7. ±0.45195.
8. 339.86.
9. 0.022709.
10. 1.1184.
11. −1.4665.
12. 0.91032.
13. $x = 3.0484$, $y = 2.0484$.
14. 8.1720.
15. 0.087963.
16. 1.2600.
17. −0.29105.
18. 0.77928.
19. 0.83944.
20. 0.96850.
21. 17.677 (18) yrs.
22. 4.6001, 4.9149, −47.121, −0.0097306, 4.5196.

Exercise IX

1. 1.9459. **2.** 4.5950. **3.** 3.6109.

4. 0, 0.69315, 1.0986, 1.3863, 1.6094, 1.7918, 1.9459, 2.0794, 2.1972, 2.3026.

5. − 0.32465. **9.** 0.49156. **12.** 126.12.

6. − 1.7354. **10.** 4.9168. **13.** 51.535.

7. − 90.006. **11.** − 2.4770. **14.** 1.3736.

8. − 0.76206.

Exercise X

1. 90°, 60°, 30°, 45°, 300°, 210°, 225°, 270°, 330°.

2. $\frac{\pi}{15}$, $\frac{\pi}{12}$, $\frac{5\pi}{12}$, $\frac{\pi}{4}$, 3.1626, $\frac{7\pi}{4}$, 0.43110.

3. 0.63638, 2.0606, 3.8331, 5.8528, 0.70814.

4. 161° 20′ 30″, 171° 53′ 00″, 102° 06′ 00″, 53° 54′ 54″, 3° 53′ 07″.

6. 127° 37′ 41″, 356° 93′ 27″.

7. 36° 49′ 06″, 53° 10′ 54″, 90°. **8.** $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$.

9. $27\frac{3}{11}$ min., $38\frac{2}{11}$ min., and $49\frac{1}{11}$ min. after 2 o'clock.

11. 3′ 22.5″.

Exercise XI

1. 6° 57′.

2. 6085.2 ft.

3. 2° 25′ 46″ (2° 26′).

4. 44.548 (44.5) ft.

5. 864,760 (865,000) miles.

6. 238,960 (239,000) miles.

7. 13,407 ft.

8. 65° 24′ 30″.

9. 182.42 (182.5) ft.

10. 304.26 (304) ft.

11. 72 yds.

Exercise XII

1. $\frac{5}{13}$, $\frac{12}{13}$, $\frac{5}{12}$, $\frac{12}{5}$, $\frac{13}{12}$, $\frac{13}{5}$.

2. $\frac{4}{5}$, $\frac{3}{5}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{5}{3}$, $\frac{5}{4}$.

3. $\frac{24}{25}$, $\frac{7}{24}$, $\frac{24}{7}$, $\frac{25}{24}$, $\frac{25}{7}$.

4. $\frac{3}{5}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{4}{3}$, $\frac{5}{3}$.

5. $\frac{8}{17}$, $\frac{8}{15}$, $\frac{15}{8}$, $\frac{17}{15}$, $\frac{17}{8}$.

6. $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}, \frac{7}{24}, \frac{25}{7}, \frac{25}{24}$.
7. 10°. **8.** − 180°. **9.** − 21°. **10.** $\frac{629}{7}$°.

Exercise XIII

1. (*a*) 0.16080, 0.87191, 0.99103, 0.68607.
(*b*) 0.98699, 0.48965, 0.13362, 0.72754.
2. 159° 18′ 34″.
3. 1.24562, 0.98908, 1.44494, 0.14736.
4. (1) $B = 21°$, $b = 6.5$, $c = 18$.
(2) $A = 48°$, $b = 602$, $a = 669$.
(3) $B = 73°$, $a = 2.6$, $c = 8.9$.
(4) $B = 12°$, $a = 49$, $b = 10$.
5. (1) $A = 25°$, $B = 65°$.
(2) $A = 25°$, $B = 65°$.
(3) $A = 37°$, $B = 53°$.
(4) $A = 68°$, $B = 22°$.
6. 300.
7. 11.7 ft.
8. 32°, 74°.
9. 7.03 ft.
10. 43.4°.
11. 57.4°.
12. 27.6 ft./sec., 11.7 ft./sec.
13. 60.4° with vertical, 20.2 ft./sec.
14. 324 lbs., 235 lbs.

Exercise XV

		sin	cos	tan	cot	sec	csc
1.	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2\sqrt{3}}{3}$
2.	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2\sqrt{3}}{3}$
	135°	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1	$-\sqrt{2}$	$-\sqrt{2}$

		sin	cos	tan	cot	sec	csc
3.	225°	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2\sqrt{3}}{3}$
4.	300°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2\sqrt{3}}{3}$
	315°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$

5. 6. **6.** 0. **7.** 3. **8.** -1.

16. 0, $\frac{2-\sqrt{3}}{4}$, $\frac{2-\sqrt{2}}{4}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2+\sqrt{2}}{4}$, $\frac{2+\sqrt{3}}{4}$, 1, $\frac{2+\sqrt{3}}{4}$, $\frac{2+\sqrt{2}}{4}$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{2-\sqrt{2}}{4}$, $\frac{2-\sqrt{3}}{4}$.

Exercise XVI

1. Functions of 30° or 150°. **5.** Functions of 150° or 210°.

	sin	cos	tan	cot	sec	csc
2.	$\pm\frac{2}{\sqrt{5}}$	$\pm\frac{1}{\sqrt{5}}$		$\frac{1}{2}$	$\pm\sqrt{5}$	$\pm\frac{\sqrt{5}}{2}$
3.	$\pm\frac{\sqrt{21}}{5}$	$-\frac{2}{5}$	$\mp\frac{\sqrt{21}}{2}$	$\mp\frac{2}{\sqrt{21}}$		$\pm\frac{5}{\sqrt{21}}$
4.	$\pm\frac{3}{\sqrt{10}}$	$\pm\frac{1}{\sqrt{10}}$	3		$\pm\sqrt{10}$	$\pm\frac{\sqrt{10}}{3}$
6.	$\pm\frac{\sqrt{8}}{3}$	$\frac{1}{3}$	$\pm\sqrt{8}$	$\pm\frac{1}{\sqrt{8}}$		$\pm\frac{3}{\sqrt{8}}$

8. $\frac{129}{220}$. **9.** $\frac{3}{10}$.

10. (*a*) 225°, (*b*) 330°, (*c*) 60°, (*d*) 90°, (*e*) 90° or 270°, (*f*) 135°, (*g*) 30° or 210°, (*h*) 45° or 135°, (*i*) 150° or 330°, (*j*) 0°, (*k*) 120° or 240°, (*l*) 60° or 120°.

Exercise XVII

1. $\sin\theta$, $\sqrt{1-\sin^2\theta}$, $\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$,

$\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$, $\dfrac{1}{\sqrt{1-\sin^2\theta}}$, $\dfrac{1}{\sin\theta}$;

$\dfrac{1}{\sqrt{1+\cot^2\theta}}$, $\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$, $\dfrac{1}{\cot\theta}$,

$\cot\theta$, $\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$, $\sqrt{1+\cot^2\theta}$;

$\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$, $\dfrac{1}{\sec\theta}$, $\sqrt{\sec^2\theta-1}$,

$\dfrac{1}{\sqrt{\sec^2\theta-1}}$, $\sec\theta$, $\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$.

2. $\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$, $\dfrac{1}{\sqrt{1+\tan^2\theta}}$, $\tan\theta$,

$\dfrac{1}{\tan\theta}$, $\sqrt{1+\tan^2\theta}$, $\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$;

$\dfrac{1}{\csc\theta}$, $\dfrac{\sqrt{\csc^2\theta-1}}{\csc\theta}$, $\dfrac{1}{\sqrt{\csc^2\theta-1}}$,

$\sqrt{\csc^2\theta-1}$, $\dfrac{\csc\theta}{\sqrt{\csc^2\theta-1}}$, $\csc\theta$.

3. ± 0.5321, ± 0.8467, 1.5911, ± 1.1811, ± 1.8793.

4. $\pm\dfrac{2ab}{a^2+b^2}$, $\pm\dfrac{2ab}{a^2-b^2}$, $\pm\dfrac{a^2-b^2}{2ab}$, $\dfrac{a^2+b^2}{a^2-b^2}$ $\pm\dfrac{a^2+b^2}{2ab}$.

5. 2.164.

Exercise XIX

3. 105,720 ft.

4. 807.3 ft.

5. 238.8 lbs.

6. 0.0051927.

7. 49.751 lbs.

8. 115.75 lb.-ins.

Exercise XX

1. $A = 31° 33' 06''$, $B = 58° 26' 54''$, $c = 757.26$.
2. $A = 41° 03' 34''$, $B = 48° 56' 26''$, $b = 74.080$.
3. $A = 62° 50' 46''$, $b = 18.168$, $c = 39.809$.
4. $a = 388.25$, $b = 548.90$, $B = 54° 43' 35''$.
5. $a = 1619.6$, $b = 42{,}535$, $A = 2° 10' 50''$.
6. $a = 388.26$, $B = 54° 43' 34''$, $A = 35° 16' 26''$.
7. $a = 125.94$, $B = 1° 01' 00''$, $c = 125.96$.
8. $b = 0.18103$, $B = 64° 41' 12''$, $c = 0.20026$.
9. $A = 74° 09' 05''$, $B = 15° 50' 55''$, $c = 9.0220$.
10. $B = 61° 12' 15''$, $b = 335.28$, $c = 382.60$.
11. $28° 57' 16''$.
12. $56° 18' 36''$.
13. 119.63 (120) ft.
14. 4.9472 ft.
15. 718.50 (718.5) ft.
16. 3.5342 in., 6.4794 in., 8.2466 in.
17. 202° 30′, 101° 15′.
18. 1718.8 (1720) ft.
19. 18.716 (18.7) miles, N 61° 21′ 48″ (61° 22′) E.
20. 9.4448 (9.5) knots, 14.135 (14.1) knots.
21. 14,731 (14,730) ft.
22. 43.166 (43.2) ft.
23. 5.147 (5.1) ft.
24. 2111.6 (2112) sq. yds.
25. 49.326 (49.3) ft.
26. 73.422 (73.4) ft., 56.619 (56.6) ft.
27. 3 : 01 : 21, 15.514 (15.5) mi.
28. 190.05 (190) mi., 65.440 (65.4) mi.
29. 348° 30′ 49″ (348° 31′), 15 hr. 54 m.
30. 144° 41′ 46″ (144° 42′), 285.50 (285.5) mi., 11.896 (11.9) knots.

Exercise XXI

3. -1.
4. 3.
5. $\tan x$.
7. $\sqrt{3} - \frac{3}{2}$.
8. 0.

Exercise XXII

3. (*a*) 7° 33′ 34″, 172° 26′ 26″.
(*b*) 105° 34′ 40″, 254° 25′ 20″.
(*c*) 117° 49′ 10″, 297° 49′ 10″.
4. − tan 11°, cot 42°, cos 45°, − sin 5°, − sec 22°.
5. − 0.66913, − 0.42262, − 0.30902, − 1.0514, 2.6050.
6. 0.43388, 1.7321, − 2, $\frac{1}{2}$.
7. − 15.819. **12.** 0.63444. **14.** 69° 50′ 39″.
8. 0.64887. **13.** − 3.4992. **15.** − 1.9679.
11. 0.10795.

Exercise XXIII

1. 8.73982 − 10, 8.39932 − 10.
2. 2.29489, 7.70511 − 10.
3. 7.79313 − 10.
4. 2.44636.
5. 7.43763 − 10.
6. 2.27223, (−) 2.27223.
7. (−) 2.24921.
8. 179° 38′ 39″.
9. 90° 44′ 55″.
10. 0° 04′ 07″, 179° 55′ 53″.
11. 0° 06′ 26″, 179° 53′ 34″.
12. 90° 26′ 36″.

Exercise XXV

1.

15°	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$2-\sqrt{3}$
	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
75°	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$2+\sqrt{3}$
	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
105°	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{1-\sqrt{3}}{2\sqrt{2}}$	$-2-\sqrt{3}$
	$\sqrt{3}-2$	$-\sqrt{6}-\sqrt{2}$	$\sqrt{6}-\sqrt{2}$

2. $-\frac{33}{56}, -\frac{16}{65}, \frac{63}{65}$.

4. $\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 3, \frac{1}{3}, \sqrt{10}, \frac{\sqrt{10}}{3}$.

5. $\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$.

27. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

28. $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

29. $\frac{1}{2}\sqrt{2-\sqrt{2}}, \quad \frac{1}{2}\sqrt{2+\sqrt{2}}, \quad \sqrt{2}-1, \quad \sqrt{2}+1, \quad \sqrt{4-2\sqrt{2}}, \quad \sqrt{4+2\sqrt{2}}$.

30. See 1, Ex. XXV.

31. $\frac{7}{9}, \frac{23}{27}, \frac{56\sqrt{2}}{17}$.

32. $\frac{4}{3}, \frac{11}{2}, \frac{2}{5\sqrt{5}}$.

45. $\frac{5}{\sqrt{26}}, \frac{24}{7}, -\frac{33}{56}$.

46. $\frac{16}{65}, -\frac{36}{323}$.

47. $-\frac{336}{527}, \frac{44}{125}, 2$.

Exercise XXVI

1. $2\sin 9x\cos 2x, \quad -2\sin 3x\sin x, \quad 0$.

11. $\cot\beta$.

Exercise XXVII

1. $b = 610.38, \quad c = 451.34, \quad C = 43^\circ\ 50'\ 24''$.

2. $a = 10.755, \quad b = 5.2092, \quad C = 36^\circ\ 09'\ 54''$.

3. $A = 30^\circ\ 50'\ 44'', \quad b = 461.28, \quad c = 691.45$.

4. $a = 55.145$, $b = 33.094$, $C = 30° 24' 00''$.

5. $A = 66° 06' 27''$, $a = 4372.3$, $b = 3405.3$.

6. (*a*) One, (*b*) None, (*c*) Two, (*d*) One, (*e*) None, (*f*) None.

7. $B_1 = 82° 26' 44''$, $b_1 = 447.27$, $C_1 = 60° 19' 16''$.
$B_2 = 23° 05' 16''$, $b_2 = 176.93$, $C_2 = 119° 40' 44''$.

8. $A_1 = 109° 26' 01''$, $a_1 = 778.94$, $B_1 = 50° 19' 34''$.
$A_2 = 30° 05' 09''$, $a_2 = 414.05$, $B_2 = 129° 40' 26''$.

9. Impossible.

10. $A = 23° 47' 50''$, $a = 31.672$, $C = 90°$.

11. $B = 28° 02' 10''$, $C = 27° 08' 38''$, $c = 30.168$.

12. $B_1 = 58° 16' 08''$, $C_1 = 74° 24' 22''$, $c_1 = 360.52$.
$B_2 = 121° 43' 52''$, $C_2 = 10° 56' 38''$, $c_2 = 71.062$.

13. $A_1 = 81° 39' 07''$, $a_1 = 167.64$, $C_1 = 55° 09' 21''$.
$A_2 = 11° 57' 49''$, $a_2 = 35{,}124$, $C_2 = 124° 50' 39''$.

14. $B_1 = 37° 52' 32''$, $C_1 = 104° 43' 45''$, $c_1 = 118.16$.
$B_2 = 142° 07' 28''$, $C_2 = 0° 28' 49''$, $c_2 = 1.0242$.

15. $A = 22° 19' 14''$, $C = 30° 28' 36''$, $c = 0.12655$.

Exercise XXVIII

1. $A = 59° 39' 00''$, $B = 39° 07' 36''$, $c = 265.97$.

2. $A = 74° 31' 44''$, $C = 40° 09' 44''$, $b = 275.54$.

3. $B = 67° 37' 44''$, $C = 51° 09' 16''$, $a = 220.10$.

4. $A = 55° 02' 57''$, $B = 40° 21' 03''$, $c = 285.07$.

5. $A = 23° 43' 48''$, $b = 272.40$, $C = 39° 34' 00''$.

6. $a = 7415.6$, $B = 13° 21' 50''$, $C = 28° 16' 58''$.

7. $A = 43° 30' 20''$, $B = 69° 17' 20''$, $c = 5.2154$.

8. $A = 92° 51' 28''$, $B = 22° 30' 32''$, $c = 0.53660$.

Exercise XXIX

1. 46.6°, 58°, 75.5°.
2. 50.8°, 56.6°, 72.6°.
3. 7.84.
4. 14.9.
5. 155.
6. $A = 106°$, $B = 39.8°$, $C = 34.1°$.
7. 84.8°.
8. 36, 20.1.

Exercise XXX

1. $A = 49° 11' 48''$, $B = 37° 36' 14''$, $C = 93° 11' 54''$.
2. $A = 52° 25' 44''$, $B = 59° 22' 32''$, $C = 68° 11' 48''$.
3. $A = 106° 18' 28''$, $B = 37° 54' 46''$, $C = 35° 46' 46''$.
4. $A = 62° 43' 16''$, $B = 79° 02' 20''$, $C = 38° 14' 32''$.
5. $A = 44° 42' 16''$, $B = 49° 37' 26''$, $C = 85° 40' 24''$.
6. $A = 83° 41' 48''$, $B = 59° 22' 20''$, $C = 36° 55' 44''$.
7. $A = 33° 35' 30''$, $B = 68° 42' 00''$, $C = 77° 42' 24''$.
8. $A = 27° 46' 44''$, $B = 33° 46' 52''$, $C = 118° 26' 20''$.

Exercise XXXI

2. 4.87139 − 10, 5.24669 − 10, 5.48037 − 10, 5.69539 − 10.

3. As in Ex. XXX except 4: $A = 62° 43' 12''$, $B = 79° 02' 11''$; 5: $A = 44° 42' 13''$, $B = 49° 37' 23''$, $C = 85° 40' 15''$; 6: $A = 83° 41' 53''$, $B = 59° 22' 24''$, $C = 36° 55' 42''$; 7: $C = 77° 42' 19''$; 8: $C = 118° 26' 08''$.

4. 42° 34' 35''.

5. 119° 08' 45''.

7. $A = 5° 35' 52''$, $a = 47.211$, $B = 84° 24' 08''$.

8. $A = \quad 55' 00''$, $a = 10.864$, $B = 89° 05' 00''$.

Exercise XXXII

1. $K = 224.43$, $r = 6.4577$, $R = 13.601$.
2. $K = 27338$, $r = 69.027$, $R = 159.08$.
3. $K = 472.26$, $r = 8.2945$, $R = 25.926$.
4. $K = 109640$, $r = 122.51$, $R = 395.37$.
5. $K = 127.98$, $r = 4.9226$, $R = 10.095$.
6. $K = 7.8870$, $r = 1.0383$, $R = 3.5616$.
7. $a = 12$, $b = 16$, $c = 20$, $K = 96$.
8. $a = 15.875$, $b = 19.844$, $c = 23.812$, $K = 156.26$.
9. 391.36. **10.** 35,103.

Exercise XXXIII

1. $A = 95° 02' 56''$, $B = 33° 06' 34''$, $C = 51° 50' 36''$, $r = 12.676$, $K = 1018.2$, $R = 34.493$.
2. $B_1 = 56° 43' 14''$, $C_1 = 90° 58' 09''$, $c_1 = 58.606$.
$B_2 = 123° 16' 46''$, $C_2 = 24° 24' 37''$, $c_2 = 24.224$.
3. $A = 23° 13' 28''$, $B = 18° 58' 14''$, $b = 112.28$.
4. $a = 168.58$, $B = 51° 17' 05''$, $b = 405.92$.
5. $A = 24° 20' 08''$, $B = 62° 41' 04''$, $C = 92° 58' 48''$.
6. $A = 39° 55' 49''$, $B = 70° 35' 57''$, $c = 42.940$.
7. $B_1 = 83° 18' 00''$, $C_1 = 65° 12' 46''$, $c_1 = 1114.6$.
$B_2 = 96° 42' 00''$, $C_2 = 51° 48' 46''$, $c_2 = 965.02$.
8. $B = 13° 25' 35''$, $C = 27° 06' 11''$, $c = 1258.2$.
9. $A = 47° 49' 34''$, $B = 58° 10' 20''$, $C = 74° 00' 00''$.
10. 26.034 (26) nautical miles, 2nd from 1st 302° 54′ 12″ (303°), 1st from 2nd 122° 54′ 12″ (123°).
11. 91.506, 59° 42′ 08″, 120° 17′ 52″.
12. 37,671 (37,700) sq. ft.
13. 271.74 (272) ft.
14. 8.0126 (8) sea miles.
15. 295.12 (295) ft.
16. 1542.8 (1543) ft.
17. 468.93 (469) ft.
18. 1202.7 (1203) ft.
19. 427.09 (427) ft.
20. 1137.4 ft.
21. 365.76 (365.8) ft.
22. 673.66 (674) ft.
23. 410.90 (411) ft.
24. 745.40 ft.
25. 7.7152 (7.72) sq. ft.

26. 41.824 (41.8) ft., 39.791 (39.8) ft.
27. 1004.7 (1005) sq. ft., 113.67 (113.7) ft.
28. 5119.5 (5120) ft.
29. 21,336 sq. ft.
30. 72.242 (72.2) ft.
31. 195.94 (195.9) ft.
32. 7.6078 (7.6) miles.
33. 90.062 (90.1) ft.
34. 1282.1 ft.
35. 140,310 sq. ft.

Exercise XXXIV

1. $n\pi + (-1)^n \frac{\pi}{6}$.
2. $n\pi + (-1)^n \frac{\pi}{2}$.
3. $2n\pi \pm \frac{3\pi}{4}$.
4. $2n\pi \pm \frac{3\pi}{4}$.
5. $n\pi + \frac{\pi}{6}$.
6. $n\pi + (-1)^n \frac{7\pi}{6}$.
7. $2n\pi$.
8. $n\pi + \frac{3\pi}{4}$.
9. $n\pi + (-)^n \frac{3\pi}{2}$.
10. $n\pi + \frac{\pi}{2}$.
11. $n\pi + \frac{\pi}{2}$.
12. $n\pi \pm \frac{\pi}{6}$.
13. $n\pi \pm \frac{\pi}{4}$.
14. $n\pi \pm \frac{\pi}{6}$.
15. $n\pi \pm \frac{\pi}{3}$.
16. $n\pi \pm \frac{\pi}{6}$.
17. $2n\pi \pm \frac{\pi}{2}$.
18. $n\pi \pm \frac{\pi}{4}$.
19. $n\pi \pm \frac{\pi}{6}$.
20. $n\pi \pm \frac{\pi}{4}$.
21. $n\pi + \frac{3\pi}{2}$.
22. $2n\pi + \frac{11\pi}{6}$.
23. $2n\pi + \frac{7\pi}{4}$.

Exercise XXXV

1. $2n\pi \pm \frac{\pi}{2}$, $n\pi + (-1)^n \frac{\pi}{6}$; 30° 90°, 150°, 270°

2. $2n\pi + \frac{\pi}{2}$; 90°.

3. $2n\pi \pm \frac{\pi}{3}$, $(2a + 1)\pi$; 60°, 180°, 300°.

4. $n\pi \pm \frac{\pi}{4}$; 45°, 135°, 225°, 315°.

5. $n\pi + (-1)^n \frac{5\pi}{4}$; 225°, 315°.

6. $n\pi$; 0°, 180°.

7. $\frac{n\pi}{4}$; 0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°.

8. $n\cdot 360° + 80° 32' 16'' \pm 70° 48' 16''$; 151° 20' 32'', 9° 44' 00''.

9. $n\pi + (-1)^n \frac{7\pi}{6}$; 210°, 330°.

10. $n\pi + (-1)^n \frac{\pi}{6}$; 30°, 150°.

11. $\frac{n\pi}{3} + \frac{\pi}{12}$; 15°, 75°, 135°, 195°, 255°, 315°.

12. $n\pi \pm \frac{\pi}{6}$; 30°, 150°, 210°, 330°.

13. $n\pi$, $2n\pi \pm \frac{\pi}{3}$; 0°, 60°, 180°, 300°.

14. $\frac{n\pi}{3} + \frac{\pi}{6}$; 30°, 90°, 150°, 210°, 270°, 330°.

15. $(2n+1)\pi$, $2n\pi \pm \cos^{-1}\frac{5}{8}$; 51° 19' 04'', 180°, 308° 40' 56''.

16. $\frac{n\pi}{3}$, $\frac{(2n+1)\pi}{12}$; 0°, 15°, 45°, 60°, 75°, 105°, 120°, 135°, 165°, 180°, 195°, 225°, 240°, 255°, 285°, 300°, 315°, 345°.

17. $n\pi$, $\frac{n\pi}{2} \pm \frac{\pi}{6}$; 0°, 30°, 60°, 120°, 150°, 180°, 210°, 240°, 300°, 330°.

18. $\frac{n\pi}{7} + \frac{\pi}{14}$; $\frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}$, etc. . . . , $\frac{27\pi}{14}$.

19. $n\pi \pm \frac{\pi}{4}$, $2n\pi \pm \frac{2\pi}{3}$; 45°, 120°, 135°, 225°, 240°, 315°.

20. $2n\pi \pm \frac{2\pi}{3}$; 120°, 240°.

21. $\frac{n\pi}{4}$, $\frac{2n\pi}{3} \pm \frac{\pi}{9}$; 0°, 20°, 45°, 90°, 100°, 135°, 140°, 180°, 220° 225°, 260°, 270°, 315°, 340°.

22. $\frac{n\pi}{5}$, $n\pi \pm \frac{\pi}{6}$; 0°, 30°, 36°, 72°, 108°, 144°, 150°, 180°, 210°, 216°, 252°, 288°, 324°, 330°.

23. $n\pi$, $2n\pi + \frac{\pi}{2}$; 0°, 90°, 180°.

24. $n.180° \pm 30°$, $n \cdot 180° \pm 60°$.

27. $\frac{2n\pi}{3}$. **29.** $n\pi - \frac{\pi}{4}$. **31.** $\frac{n\pi}{2} + (-1)^n \frac{7\pi}{12}$.

Exercise XXXVI

(a positive) **2.** $\pi - \sin^{-1}\sqrt{1-a^2} =$

$$\cos^{-1}(-a) = \tan^{-1}\frac{-\sqrt{1-a^2}}{a} = \cot^{-1}\frac{-a}{\sqrt{1-a^2}} =$$

$$\pi - \csc^{-1}\frac{1}{\sqrt{1-a^2}} = \sec^{-1}\left(-\frac{1}{a}\right).$$

3. $\sin^{-1}\frac{1}{\sqrt{1+a^2}} = \cos^{-1}\frac{a}{\sqrt{1+a^2}} = \tan^{-1}\frac{1}{a} =$

$$\cot^{-1}a = \csc^{-1}\sqrt{1+a^2} = \sec^{-1}\frac{\sqrt{1+a^2}}{a}.$$

4. $\pi - \sin^{-1}\frac{\sqrt{a^2-1}}{a} = \cos^{-1}\left(-\frac{1}{a}\right) =$

$$\tan^{-1} - \sqrt{a^2-1} = \cot^{-1}\frac{-1}{\sqrt{a^2-1}} =$$

$$\pi - \csc^{-1}\frac{a}{\sqrt{a^2-1}} = \sec^{-1}(-a).$$

5. $\sin^{-1}\left(\pm\frac{1}{a}\right) = \cos^{-1}\frac{\pm\sqrt{a^2-1}}{a} =$

$$\tan^{-1}\frac{1}{\sqrt{a^2-1}} = \cot^{-1}\sqrt{a^2-1} =$$

$$\csc^{-1}(\pm a) = \sec^{-1}\frac{\pm a}{\sqrt{a^2-1}}.$$

6. $\cos^{-1}(1-a) = \tan^{-1}\frac{\sqrt{2a-a^2}}{1-a} = \cot^{-1}\frac{1-a}{\sqrt{2a-a^2}} =$

$$\left\{\sin^{-1}\sqrt{2a-a^2} = \csc^{-1}\frac{1}{\sqrt{2a-a^2}} \quad \text{when} \quad a < 1\right\} =$$

$$\left\{\pi - \sin^{-1}\sqrt{2a-a^2} = \pi - \csc^{-1}\frac{1}{\sqrt{2a-a^2}} \text{ when } a > 1\right\}.$$

Exercise XXXVII

1. If $a > b$, $\cos^{-1} a - \cos^{-1} b = -\cos^{-1}(ab + \sqrt{1-a^2}\sqrt{1-b^2})$.

2. If $b > a$, $\tan^{-1} a - \tan^{-1} b = \tan^{-1}\dfrac{a-b}{1+ab} - \pi$.

3. If $a > \dfrac{1}{\sqrt{2}}$, $2\sin^{-1} a = \pi - \sin^{-1}(2a\sqrt{1-a^2})$.

6. $\tan^{-1} a + \tan^{-1} b = \pi + \tan^{-1}\dfrac{a+b}{1-ab}$, otherwise same.

7. $2\cos^{-1} a = 2\pi - \cos^{-1}(2a^2 - 1)$.

Exercise XXXVIII

19. $\dfrac{\pi}{4}$.

Exercise XXXIX

1. No solution for principal values.
2. 0, $\sqrt{1\frac{1}{2}}$.
3. No solution for principal values.
4. $\frac{2}{9}$, $-\frac{2}{3}$.
5. ± 1, $\pm(\sqrt{2}+1)$.
6. 13.
7. 1.
8. $\sqrt{3}$.
9. $\pm\frac{1}{3}\sqrt{5}$.
10. $\frac{4}{3}$.

Exercise XLII

1. 2, 60° or −2, 240°; 2, 240° or −2, 60°; $2\sqrt{2}$, 315° or $-2\sqrt{2}$, 135°; $2\sqrt{2}$, 135° or $-2\sqrt{2}$, 315°; 13, 67° 22′ 46″ or −13,247° 22′ 46″; 5, 126° 52′ 12″ or −5, 306° 52′ 12″.

2. $\left(\dfrac{3}{2}, \dfrac{3\sqrt{3}}{2}\right)$, (−3.3547, 2.1786), (−2.7351, 6.4436), $\left(-\dfrac{7}{2}, \dfrac{7\sqrt{3}}{2}\right)$, (−6.9882, 26.080).

3. Circle of radius 10 with center at origin; straight line inclined 30° to the horizontal.

Exercise XLIII

1. $\sqrt{13}(\cos 236° 18' 35'' + j \sin 236° 18' 35'') = \sqrt{13}\,\underline{/123° 41' 25''}$.
2. $\sqrt{41}(\cos 38° 39' 34'' + j \sin 38° 39' 34'') = \sqrt{41}\,\underline{/38° 39' 34''}$.
3. $\sqrt{2}\,(\cos 315° + j \sin 315°) = \sqrt{2}\,\underline{/-45°}$.
4. $1.7\,(\cos 61° 55' 41'' + j \sin 61° 55' 41'') = 1.7\,\underline{/61° 55' 41''}$.
5. $18.949\,(\cos 4° 06' 57'' + j \sin 4° 06' 57'') = 18.949\,\underline{/4° 06' 57''}$.
6. $\frac{5}{2} - \frac{5\sqrt{3}}{2}j$.
7. $-\sqrt{2} + \sqrt{2}\,j$.
8. $-4j$.
9. $\frac{5\sqrt{3} - 5}{2\sqrt{2}} + \frac{5\sqrt{3} + 5}{2\sqrt{2}}j$.
10. $\frac{1}{2}\sqrt{2 + \sqrt{2}} + \frac{1}{2}\sqrt{2 - \sqrt{2}}\,j$.
11. $2.19405 + 2.04600j$.
12. $-\frac{\sqrt{3}}{2} - \frac{1}{2}j$.
13. $j, -1, -j, 1, j$, etc.; 0.
14. $x = 5,\ y = -3$.
15. $x = -\frac{13}{19},\ y = -\frac{10}{19}$.
16. No real values.

Exercise XLIV

1. $7 + 7j$.
2. $2 - 3j$.
3. $7 + 6j$.
4. $14 + 38j$.
5. $6 + 9j$.

Exercise XLV

1. $\sqrt{2}\,\underline{\angle 75°}$ or $\frac{\sqrt{3} - 1}{2} - j\,\frac{\sqrt{3} + 1}{2}$.
2. $\underline{42/90°}$ or $42j$.
3. $\underline{0.5/90°}$ or $\frac{1}{2}j$.
4. $\underline{0.5/150°}$ or $-\frac{\sqrt{3}}{4} - \frac{1}{4}j$.
5. $\underline{2.9833/39° 33' 25''}$ or $\frac{1}{130}(299 + 247j)$.

7. $16\angle 120°$ or $-8+8\sqrt{3}j$.
8. $10^4\angle 147°\ 28'\ 46''$ or $-8432+5376j$.
9. $8\sqrt{2}\angle 45°$ or $8-8j$.
10. $1682\cos 87°\ 12'\ 20''$ or 82.

Exercise XLVI

1. $\sqrt[6]{2}\angle 15°$, $\sqrt[6]{2}\angle 135°$, $\sqrt[6]{2}\angle 105°$.
2. $1\angle 60°$, $1\angle 132°$, $1\angle 156°$, $1\angle 84°$, $1\angle 12°$.
3. $\sqrt[4]{145}\angle 159°\ 11'\ 00''$, $\sqrt[4]{145}\angle 20°\ 49'\ 00''$.
4. $\sqrt[14]{18}\angle 45°$, $\sqrt[14]{18}\angle 96°\ 25'\ 43''$, $\sqrt[14]{18}\angle 147°\ 51'\ 26''$, $\sqrt[14]{18}\angle 160°\ 42'\ 51''$, $\sqrt[14]{18}\angle 109°\ 17'\ 08''$, $\sqrt[14]{18}\angle 57°\ 51'\ 25''$, $\sqrt[14]{18}\angle 6°\ 25'\ 42''$.
5. $1, -\frac{1}{2}+\frac{\sqrt{3}}{2}j, -\frac{1}{2}-\frac{\sqrt{3}}{2}j$.
6. $\frac{1}{2}+\frac{\sqrt{3}}{2}j, -1, \frac{1}{2}-\frac{\sqrt{3}}{2}j$.
7. $2, 2\angle 72°, 2\angle 144°, 2\angle 144°, 2\angle 72°$.
8. $\pm 1, \pm j$.
9. $2, 1+\sqrt{3}j, -1+\sqrt{3}j, -2, -1-\sqrt{3}j, 1-\sqrt{3}j$.
10. $1\angle 22.5°$, $1\angle 67.5°$, $1\angle 112.5°$, $1\angle 157.5°$, $1\angle 157.5°$, $1\angle 112.5°$, $1\angle 67.5°$, $1\angle 22.5°$.

Exercise XLVII

1. $2\sin\theta\cos\theta$, $\cos^2\theta-\sin^2\theta$.
2. $4\sin\theta\cos^3\theta-4\cos\theta\sin^3\theta$, $\cos^4\theta-6\sin^2\theta\cos^2\theta+\sin^4\theta$.
3. $\sin^5\theta-10\sin^3\theta\cos^2\theta+5\sin\theta\cos^4\theta$, $\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$.
4. $6\cos^5\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta$, $\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta$.
5. $-\frac{1}{4}\sin 3\theta+\frac{3}{4}\sin\theta$.

6. $\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.
7. $\frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$.
8. $\frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.
9. $\frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.
10. $\frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$.
11. $-\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$.
12. $\frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$.

Exercise XLVIII

3, 5. 40° 16′ 00″.
4. 64° 06′ 34″.
6. 64° 23′ 24″.
7. 68° 22′ 00″.
8. 53° 19′ 26″ or 126° 40′ 34″.
9. 19° 58′ 37″.
10. $a = 42°\ 45'\ 00''$, $B = 64°\ 26'\ 10''$, $C = 77°\ 21'\ 40''$.

Exercise XLIX

1. $A = 67°\ 47'\ 32''$, $B = 49°\ 06'\ 24''$, $b = 45°\ 00'\ 00''$.
2. $A = 55°\ 16'\ 00''$, $B = 114°\ 35'\ 36''$, $c = 108°\ 30'\ 02''$.
3. $A = 35°\ 20'\ 00''$, $a = 34°\ 16'\ 23''$, $b = 106°\ 00'\ 01''$.
4. $A = 140°\ 00'\ 00''$, $B = 80°\ 00'\ 00''$, $c = 102°\ 07'\ 50''$.
5. $a = 52°\ 41'\ 43''$, $b = 65°\ 19'\ 16''$, $c = 75°\ 20'\ 30''$.
6. $a = 66°\ 45'\ 52''$, $B = 117°\ 28'\ 48''$, $c = 101°\ 12'\ 21''$.
7. $a = 123°\ 22'\ 41''$, $A = 119°\ 31'\ 57''$, $B = 63°\ 38'\ 00''$.
8. $B_1 = 26°\ 01'\ 35''$, $b_1 = 24°\ 13'\ 19''$, $c_1 = 69°\ 13'\ 51''$.
$B_2 = 153°\ 58'\ 25''$, $b_2 = 155°\ 46'\ 41''$, $c_2 = 110°\ 46'\ 09''$.
9. $a = 54°\ 32'\ 34''$, $b = 51°\ 43'\ 38''$, $c = 68°\ 56'\ 32''$.
10. $B_1 = 124°\ 15'\ 53''$, $b_1 = 133°\ 40'\ 20''$, $c_1 = 61°\ 04'\ 00''$.
$B_2 = 55°\ 44'\ 07''$, $b_2 = 46°\ 19'\ 40''$, $c_2 = 118°\ 56'\ 00''$.
11. $B_1 = 115°\ 16'\ 00''$, $b_1 = 143°\ 10'\ 00''$, $c_1 = 41°\ 31'\ 17''$.
$B_2 = 64°\ 44'\ 00''$, $b_2 = 36°\ 50'\ 00''$, $c_2 = 138°\ 28'\ 43''$.
12. $B_1 = 68°\ 56'\ 48''$, $b_1 = 65°\ 42'\ 50''$, $c_1 = 102°\ 23'\ 40''$.
$B_2 = 111°\ 03'\ 12''$, $b_2 = 114°\ 17'\ 10''$, $c_2 = 77°\ 36'\ 20''$.
13. $A = 100°$, $a = 112°$, $b = 25°\ 52'\ 34''$.

14. $A_1 = 27° 36' 59''$, $a_1 = 25° 52' 34''$, $c_1 = 70° 18' 10''$.
$A_2 = 152° 23' 01''$, $a_2 = 154° 07' 26''$, $c_2 = 109° 41' 50''$.
15. $A = 79° 30' 26''$, $B = 152° 04' 32''$, $c = 110° 27' 14''$.
16. $A_1 = 100°$, $a_1 = 112°$, $c_1 = 70° 17' 50''$.
$A_2 = 80°$, $a_2 = 68°$, $c_2 = 109° 42' 10''$.
17. $a = 16°$, $b = 116°$, $c = 114° 55' 20''$.
18. $A = 57° 16' 20''$, $B = 60° 47' 24''$, $c = 68° 56' 29''$.
19. $a = b = 32° 38' 22''$, $c = 39° 54' 28''$.
20. $c = 23° 16' 40''$, $C = 30° 17' 16''$.

Exercise L

1. $A = 90°$, $b = 60°$, $c = 90°$.
2. $a = 90°$, $b = 103°$, $c = 90°$.
3. $b_1 = 10° 28' 54''$, $B_1 = 8° 19' 16''$, $C_1 = 127° 17' 48''$.
$b_2 = 169° 31' 06''$, $B_2 = 171° 40' 44''$, $C_2 = 52° 42' 12''$.
4. $A = 112° 51' 56''$, $C = 69° 14' 54''$, $b = 26° 02' 41''$.
5. $a = 81° 48' 47''$, $b = 161° 19' 33''$, $C = 64° 48' 26''$.
6. $B = 40° 16' 38''$, $a = 150° 10' 17''$, $b = 67° 41' 55''$.
7. $b_1 = 53° 30' 26''$, $B_1 = 42° 24' 13''$, $C_1 = 122° 59' 08''$.
$b_2 = 126° 29' 34''$, $B_2 = 137° 35' 47''$, $C_2 = 57° 00' 52''$.
8. $a = 86° 55' 00''$, $b = 41° 28' 22''$, $B = 41° 22' 38''$.

Exercise LI

12. $A = 119° 45' 15''$, $E = 114° 56' 06''$.
13. $B = 40° 43' 13''$, $C = 110° 26' 55''$.
14. $a = 137° 22' 42''$, $c = 19° 58' 36''$.
15. $A = 95° 48' 14''$, $B = 97° 34' 08''$, $C = 66° 44' 44''$.
16. $a = 84° 21' 56''$, $b = 82° 33' 31''$.
23. $a = 168° 15' 42''$, $A = 169° 38' 45''$, $b = 95° 33' 38''$.
24. $A = 86° 55' 00''$, $B = 41° 29' 49''$, $b = 41° 24' 14''$.

Exercise LII

1, 2. $a = 80° 35' 45''$, $B = 126° 00' 31''$, $C = 59° 08' 52''$.
3. $a = 64° 23' 24''$, $B = 85° 37' 51''$, $C = 67° 24' 41''$.

4. $a = 86° 41' 49''$, $B = 50° 25' 00''$, $C = 130° 23' 26''$.
5. $a = 111° 10' 32''$, $B = 21° 42' 13''$, $C = 61° 09' 40''$.
6. $a = 41° 20' 16''$, $B = 146° 16' 46''$, $C = 61° 34' 32''$.
7. $a = 64° 23' 15''$.
8. $a = 96° 20' 41''$, $B = 45° 04' 42''$, $C = 56° 16' 14''$.

Exercise LIII

3, 4, 5. $a = 40° 01' 05''$, $b = 38° 31' 05''$, $C = 130° 03' 48''$.
6, 7. $A = 60° 17' 38''$, $b = 80° 38' 28''$, $c = 104° 28' 13''$.
8, 9. $a = 117° 07' 36''$, $B = 97° 59' 30''$, $c = 147° 02' 33''$.
10. $a = 114° 46' 56''$, $b = 130° 32' 09''$, $C = 59° 49' 08''$.
11. $a = 114° 52' 21''$, $b = 135° 57' 39''$, $C = 96° 49' 36''$.

Exercise LIV

2. $B_1 = 42° 37' 24''$, $c_1 = 153° 39' 10''$, $C_1 = 160° 01' 40''$.
$B_2 = 137° 22' 36''$, $c_2 = 90° 05' 10''$, $C_2 = 50° 19' 05''$.
3. $B_1 = 112° 25' 55''$, $c_1 = 52° 19' 57''$, $C_1 = 77° 36' 06''$.
$B_2 = 67° 34' 05''$, $c_2 = 16° 52' 57''$, $C_2 = 20° 59' 52''$.
4. $B = 63° 01' 40''$, $C = 84° 50' 28''$, $c = 80° 51' 28''$.
5. $B = 23° 08' 20''$, $C = 149° 02' 17''$, $c = 110° 22' 40''$.
6. Impossible.
7. $B_1 = 80° 22' 32''$, $C_1 = 82° 38' 22''$, $c_1 = 62° 26' 42''$.
$B_2 = 99° 37' 28''$, $C_2 = 60° 12' 34''$, $c_2 = 50° 52' 42''$.
8. $B_1 = 59° 24' 17''$, $C_1 = 115° 39' 51''$, $c_1 = 97° 33' 15''$.
$B_2 = 120° 35' 43''$, $C_2 = 27° 00' 01''$, $c_2 = 29° 59' 15''$.
9. $B_1 = 75° 37' 30''$, $C_1 = 166° 04' 36''$, $c_1 = 166° 09' 44''$.
$B_2 = 104° 22' 30''$, $C_2 = 91° 21' 28''$, $c_2 = 96° 26' 20''$.
10. $B_1 = 116° 26' 30''$, $C_1 = 166° 13' 06''$, $c_1 = 164° 59' 18''$.
$B_2 = 63° 33' 30''$, $C_2 = 56° 17' 46''$, $c_2 = 64° 45' 18''$.

Exercise LV

2. $b_1 = 58° 58' 40''$, $c_1 = 117° 32' 43''$, $C_1 = 94° 18' 33''$.
$b_2 = 121° 01' 20''$, $c_2 = 41° 54' 19''$, $C_2 = 48° 41' 33''$.

3. $b_1 = 42° 38' 33''$, $c_1 = 129° 39' 31''$, $C_1 = 89° 53' 25''$.
$b_2 = 137° 21' 27''$, $c_2 = 19° 59' 45''$, $C_2 = 26° 22' 11''$.
4. $a = 125° 22' 40''$, $c = 155° 48' 12''$, $C = 155° 50' 58''$.
5. $a_1 = 42° 34' 20''$, $c_1 = 129° 41' 41''$, $C_1 = 89° 51' 33''$.
$a_2 = 137° 25' 40''$, $c_2 = 19° 51' 49''$, $C_2 = 26° 12' 17''$.
6. $b_1 = 128° 55' 40''$, $c_1 = 16° 32' 41''$, $C_1 = 21° 21' 48''$.
$b_2 = 51° 04' 20''$, $c_2 = 151° 14' 55''$, $C_2 = 142° 01' 28''$.
7. $b_1 = 99° 37' 28''$, $c_1 = 97° 21' 38''$, $C_1 = 117° 33' 18''$.
$b_2 = 80° 22' 32''$, $c_2 = 119° 47' 26''$, $C_2 = 129° 07' 18''$.
8. Impossible.
9. $a = 114° 26' 50''$, $c = 82° 33' 31''$, $C = 79° 10' 30''$.
10. $a_1 = 42° 37' 18''$, $c_1 = 129° 41' 05''$, $C_1 = 89° 54' 19''$.
$a_2 = 137° 22' 42''$, $c_2 = 19° 58' 36''$, $C_2 = 26° 21' 18''$.

Exercise LVI

1. $A = 20° 09' 56''$, $B = 55° 52' 32''$, $C = 114° 20' 14''$.
2. $A = 22° 04' 46''$, $B = 124° 17' 16''$, $C = 61° 29' 40''$.
3. $A = 97° 10' 56''$, $B = 47° 16' 42''$, $C = 59° 20' 36''$.
4. $A = 73° 37' 25''$, $B = 65° 42' 35''$, $C = 81° 15' 00''$.
5. $A = 110° 51' 18''$, $B = 48° 56' 06''$, $C = 38° 26' 46''$.
6, 7, 8. $a = 40° 05' 26''$, $b = 118° 22' 07''$, $c = 153° 38' 42''$.
9. $a = 101° 30' 00''$, $b = 50° 45' 00''$, $c = 61° 45' 00''$.
10. $a = 48° 33' 37''$, $b = 43° 18' 29''$, $c = 61° 34' 12''$.
11. $a = 101° 33' 24''$, $b = 80° 26' 04''$, $c = 71° 24' 26''$.

Exercise LVII

1. $a = 49° 34' 00''$, $B = 87° 08' 14''$, $b = 109° 14' 00''$.
2. $c_1 = 164° 59' 18''$, $B_1 = 63° 33' 30''$, $C_1 = 166° 13' 06''$.
$c_2 = 64° 45' 18''$, $B_2 = 116° 26' 30''$, $C_2 = 56° 17' 46''$.
3. $C = 72° 02' 09''$, $b = 83° 12' 16''$, $a = 31° 40' 19''$.
4. $a = 114° 46' 56''$, $b = 130° 32' 09''$, $C = 59° 49' 08''$.
5. $C = 145° 24' 40''$, $A = 111° 45' 50''$, $b = 71° 37' 20''$.
6. $B = 71° 55' 45''$, $a = 64° 21' 15''$, $C = 111° 06' 15''$.
7. $A = 94° 28' 44''$, $B = 66° 38' 28''$, $C = 58° 42' 40''$.

8. $a = 62° 08' 00''$, $b = 80° 30' 30''$, $c = 70° 20' 30''$.
9. $A = 56° 34' 00''$, $B = 26° 22' 25''$, $c = 101° 19' 27''$.
10. $A = 145° 55' 40''$, $B = 19° 17' 47''$, $C = 38° 14' 00''$.

Exercise LVIII

1. 3164,9 geog. miles. **2.** 4760.1 naut. miles, 236° 52′ 14″.
5. 4452.1 naut. miles, 302° 58′ 56″.
6. Sides 97° 36′, 87° 11′, 61° 25′, area 17,370,000 sq. miles.
7. 6446 naut. miles, 240° 17′ 14″, 226° 43′ 52″.
8. 4835.8 naut. miles, 126° 00′ 31″. **16.** $C_n = 82° 04' 28''$
9. 3863.4 naut. miles, 274° 22′ 09″. $D = 6748.6$ miles.
10. 5201.8 naut. miles, 50° 25′ 00″. $L_v = 28° 29' 44''$ S.
11. 5780.7 naut. miles, 314° 55′ 17″. $\lambda_v = 136° 13' 45''$ E.

Exercise LIX

1. 10° 10′ N.
2. 56° 20′ S.
3. 69° 35′ N.
4. 14° 08′ N.
5. 7° 42′ S.
6. 52° 26′ S.
7. 77° 54′ N.
8. Absurd.
9. 21° 17′ S.
10. 7° 28′ N.
11. 35° 41′ N.
12. 37° 33′ N.
13. 24° 17′ N.
14. 40° 00′ N.
15. 83° 40′ S.
16. 78° 00′ N.
17. 45° 32′ N.
18. 65° 53′ N.

Exercise LX

1. $t = 4^h 37^m 46^s$, $Z =$ N 87° 16′ 22″ W.
2. $t = 3^h 00^m 56^s$, $Z =$ N 129° 05′ 40″ W.
3. $t = 2^h 46^m 48^s$, $Z =$ N 132° 55′ 06″ W.
4. $t = 4^h 41^m 52^s$, $Z =$ N 103° 14′ 45″ E.
5. $t = 4^h 09^m 24^s$, $Z =$ N 117° 39′ 10″ E.
6. $t = 4^h 36^m 46^s$, $Z =$ N 87° 13′ 50″ E.
7. (*a*) $1^h 39^m 46^s$ A.M., $10^h 20^m 14^s$ P.M.
(*b*) $4^h 50^m 59^s$ A.M., $7^h 09^m 01^s$ P.M.
(*c*) $5^h 47^m 56^s$ A.M., $6^h 12^m 04^s$ P.M.
(*d*) $4^h 41^m 44^s$ A.M., $7^h 18^m 16^s$ P.M.

8. $5^h\ 31^m\ 36^s$, $18^h\ 28^m\ 24^s$.
9. $9^h\ 49^m\ 00^s$, $14^h\ 11^m\ 00^s$.
10. (*a*) $1^h\ 43^m\ 28^s$, $11^h\ 28^m\ 26^s$, $9^h\ 04^m\ 38^s$, $1^h\ 43^m\ 28^s$.
(*b*) $2^h\ 11^m\ 45^s$, $4^h\ 41^m\ 08^s$, $14^h\ 55^m\ 22^s$, $2^h\ 11^m\ 45^s$.
(*c*) $1^h\ 36^m\ 14^s$, $8^h\ 47^m\ 32^s$, $12^h\ 00^m\ 00^s$, $1^h\ 36^m\ 14^s$.
11. $t = 2^h\ 58^m\ 59^s$, Z = N 46° 32′ 43″ W.
12. $t = 2^h\ 00^m\ 42^s$, Z = N 41° 00′ 57″ E.
13. $t = 4^h\ 01^m\ 48^s$, Z = N 51° 13′ 13″ W.

Exercise LXI

1. h = 43° 51′ 28″, Z = N 89° 20′ 04″ W.
2. h = 34° 53′ 54″, Z = N 149° 30′ 22″ W.
3. h = 27° 31′ 20″, Z = N 129° 37′ 50″ W.
4. h = 48° 52′ 06″, Z = N 128° 19′ 18″ E.
5. h = 51° 39′ 30″, Z = N 44° 40′ 42″ E.
6. h = 25° 17′ 00″, Z = N 100° 34′ 50″ W.
7. h = 2° 11′ 44″, Z = N 126° 58′ 00″ E.
8. h = 30° 00′ 00″, Z = N 46° 33′ 10″ W.
9. h = 54° 00′ 00″, Z = N 43° 04′ 35″ W.
10. h = 35° 48′ 00″, Z = N 36° 22′ 32″ W

Exercise LXII

1. $\sin t = \sec d \cos h$, $\sin L = \sin d \csc h$.
2. $t = 3^h\ 4^m\ 58^s$, L = 59° 50′ 34″ N.
3. $\cos t = -\tan L \tan d$, $\sin A = \sin d \sec L$.
4. $t = 4^h\ 29^m\ 19^s$ A.M., A = E 33° 35′ 03″ N.
5. $t = 3^h\ 48^m\ 56^s$, L = 55° 01′ 47″ S.
6. $t = 5^h\ 7^m\ 46^s$, A = E 20° 14′ 25″ S.
7. L_1 = 67° 34′ 12″ N, C_1 = 74° 17′ 52″, D_1 = 382.46 mi.
L_2 = 68° 20′ 21″ N, C_2 = 84° 20′ 35″, D_2 = 134.51 mi.
8. L_1 = 10°56′39″S, Long.$_1$ = 154°37′50″W, C_1 = 216°00′13″
L_2 = 10°56′39″S, Long.$_2$ = 41°04′50″E, C_2 = 143°59′47″
C_0 = 215° 52′ 40″.

9. $63\frac{2}{15}$ mi., $L_1 = 44° 31' 46''$ N, Long.$_1 = 177° 33' 07''$ E,
$L_2 = 44° 16' 30''$ N, Long.$_2 = 176° 07' 21''$ E.
10. 57.7 mi., 303.1 mi.
11. 297° 42′ 15″, 5991.8 mi.
12. $L_v = 46° 07' 09''$ N, Long.$_v = 163° 32' 50''$ W.
13. 224° 09′ 00″, 5831.9 mi.
14. $L_v = 54° 45' 07''$ S, Long.$_v = 123° 13' 28''$ E,
Long. Eq. $= 146° 46' 32''$ W, $C = 215° 14' 53''$.
15. $L = 6° 55' 32''$S, Long. $= 151° 42' 00''$ W, $C = 215° 32' 47''$,
Distance gone = 3106.8 mi., Distance to go = 2725.1 mi.
16. $L = 3° 37' 30''$ S, Long. $= 149° 20' 29''$ W, $C = 215° 19' 47''$,
Distance gone = 2863.5 mi., Distance to go = 2968.4 mi.
17. $L = 4° 20' 23''$ S, Long. $= 149° 50' 59''$ W, $C = 215° 21' 52''$,
Distance gone = 2916 mi., Distance to go = 2915.9 mi.
18. $h = 34° 42' 00''$, $Az. = 244° 06' 30''$.
19. $h = 36° 40' 18''$, $Az. = 94° 00' 24''$.
20. $h = 21° 36' 08''$, $Az. = 87° 14' 34''$.
21. $Az. = 75° 12' 28''$.
22. $L_1 = 39° 15' 46''$ N, Long.$_1 = 57° 23' 21''$ W.
$L_2 = 33° 23' 28''$ N, Long.$_2 = 34° 37' 06''$ W.
23. $L_v = 66° 10' 02''$ N, Long.$_v = 167° 34' 16''$ E.
24. $3^h\ 51^m\ 33^s$ P.M.
25. $4^h\ 51^m\ 28^s$ A.M., $7^h\ 8^m\ 32^s$ A.M.
26. $2^h\ 51^m\ 47^s$ A.M., $9^h\ 8^m\ 13^s$ A.M.
27. $2^h\ 54^m\ 13^s$ A.M., $9^h\ 5^m\ 47^s$ P.M.
28. 99° 57′ 30″.
29. $p = 57° 52' 27''$, $t = 8^h\ 2^m\ 10^s$.
30. $Az. = 121° 07' 11''$.

Exercise LXV

2. $AB = 49°$, $BZ_r = 41°$.
3. 64°, 116°.
4. 74°, 106°.
5. 45°, 71°, 64°.

Exercise LXVI

1. $a = 48° 45', b = 37° 45', C = 92° 45'.$
2. $a = 48° 45', B = 54° 30', C = 86° 30'.$
3. $B_1 = 60°, C_1 = 107° 30', c_1 = 72° 30', B_2 = 120°, C_2 = 40° 15', c_2 = 40° 15'.$
4. $a_1 = 118° 15', c_1 = 117° 15', C_1 = 104° 15', a_2 = 61° 45', c_2 = 23° 15', C_2 = 25° 30'.$
5. $a = 103° 45', b = 52° 30', c = 76° 15'.$
6. $B = 66° 15', C = 100° 45', c = 104° 30'.$
7. $a = 99° 45', b = 73° 15', C = 45° 45'.$
8. $A = 49° 15', B = 118° 30', C = 61° 30'.$
9. $a = 64° 30', B = 133° 15', C = 46°.$

Exercise LXVII

1. $D = 54° 30' = 3270$ miles; $c.c. = 294° 45'.$
2. $118° 30' = 7110$ miles; $289° 30'.$
3. $100° 45' = 6045$ miles; $298° 15'.$
4. $113° = 6780$ miles; $115° 45'.$
5. $107° 30' = 6450$ miles; $240° 15'.$
6. $121° 30' = 7290$ miles; $332°.$
7. $131° 45' = 7905$ miles; $242° 45'.$
8. $134° 45' = 8085$ miles; $220° 30'.$
9. $143° 30' = 8610$ miles; $324°.$
10. $48° 15' = 2895$ miles; $292° 45'.$
11. $62° = 3720$ miles; $316°.$
12. $41° = 2460$ miles; $314° 45'.$
13. $79° 30' = 4770$ miles; $240° 15'.$
14. $116° = 6960$ miles; $124° 45'.$
15. $L = 40° 15'$ N, $\lambda = 55° 45'$ W.

Exercise LXVIII

1. $t = 77°, Z = \text{N } 82° 15' \text{ W}.$
2. $h = 16° 45', Z = \text{N } 119° \text{ E}.$

3. $L_1 = 34° 30'$ S, $Z_1 =$ N $46° 30'$ E, $L_2 = 65° 30'$ N, $Z_2 =$ N $133° 30'$ E.

4. $h = 46° 15'$, $Z =$ N $99° 45'$ W.

5. $t = 46° 15'$, $Z =$ N $99°$ W.

6. $L_1 = 9° 15'$ S, $L_2 = 65° 45'$ N.

7. $h = 46° 30'$, $Z =$ N $121° 30'$ W.

8. $t = 107° 45'$, $Z =$ N $63° 30'$ W.

10. $t = 38° 45'$, $Z =$ N $126° 15'$ W.

11. $L_1 = 8° 45'$ N, $L_2 = 25° 45'$ N.

12. $h = 50° 30'$, $Z =$ N $36° 45'$ W.

13. $t = 49° 45'$, $Z =$ N $124°$ W.

14. $L_1 = 18°$ S, $Z_1 =$ N $40° 15'$ W, $L_2 = 66° 45'$ N, $Z_2 =$ N $139° 45'$ W.

15. $L_1 = 14°$ S, $L_2 = 59° 30'$ N.

16. $L = 47° 30'$ S.

17. $L_1 = 23° 15'$ S, $L_2 = 79° 30'$ N.

18. $L = 13°$ N.

19. $L = 17° 45'$ S.

20. $L_1 = 9° 15'$ S, $Z_1 =$ N $39° 45'$ E, $L_2 = 65° 45'$ N, $Z_2 =$ N $140°$ E.

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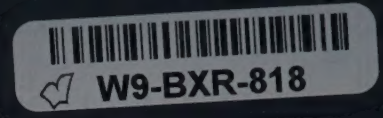
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